



(REVIEW ARTICLE)



Review of Adams-Bashforth method for numerical solution of first order ordinary differential equations

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Abstract

This project work presents a comprehensive review of the Adams-Bashforth method for the numerical solution of explicit first-order ordinary differential equations (ODEs). The study begins with a historical overview of the development of ordinary differential equations (ODEs), tracing back to the seminal works of Newton, Leibniz, and their contemporaries. The evolution of differential equations as a distinct mathematical discipline and their wide-ranging applications across various fields are discussed. Furthermore, the paper provides an in-depth analysis of the Adams-Bashforth method, a prominent numerical technique for solving ODEs. The method is derived and analyzed using Mathematica Software, demonstrating its efficacy and accuracy in approximating solutions to ODEs. A comparison with existing methods such as the Euler method and the Runge-Kutta method highlights the advantages of the Adams-Bashforth method, particularly in terms of computational efficiency and accuracy. Moreover, fundamental concepts related to ODEs, including the initial value problem, existence, and uniqueness of solutions, are explored in the context of numerical solution methods. The properties of linear multistep methods and their relevance to solving ODEs efficiently are also discussed. In conclusion, this project work provides valuable insights into the Adams-Bashforth method as a powerful tool for the numerical solution of first-order ODEs. By leveraging modern computational tools such as Mathematica Software, the method offers a practical and efficient approach to solving complex differential equations encountered in various scientific and engineering applications.

Keywords: Adams-Bashforth method; Numerical solution; First order; Equations

1. Introduction

1.1. Brief history on differential equation

Differential Equation was a term introduced by G.W. Leibniz in 1676. Leibniz and Newton, specifically, published and disseminated their differential equation solutions for the first time in 1693, marking the beginning of the differential equation as a separate branch of mathematics.

The study of differential equations, according to British mathematician Edward Ince, and other mathematics historians is said to have begun in 1675, when German mathematician Gottfried Wilhelm Leibniz wrote the following equation:

$$\int x dx = \frac{1}{2} x^2$$

In the latter part of the 17th century, these equations were studied as mechanical and geometrical mathematics. The description of the motion of solid bodies or mass points can now be reduced to solving an Ordinary Differential Equation

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(ODE) thanks to the Newton's law of mechanics. The radio-technological or satellite trajectory investigates the stability of a plane in flight and uses differential equation to describe the progression of a chemical process.

It has been required in many disciplines, including engineering, physics, chemistry, and even biology and economics, to construct abstract symbolic mathematical models that could best describe a real-world problem or system. Equations that often contain an unknown function known as the "dependent variable" and one or more derivatives are examples of such derived abstract symbolic models.

Differential equations are created through the modeling of actual systems using symbolic mathematical representations.

A differential equation can naturally describe a wide variety of occurrences in science and applied mathematics. Even electrical circuits abide by the current and derivative principles outlined by Differential Equation as time and acceleration are coupled in the dynamics.

1.2. Types of differential equations

There are basically six types of Differential Equations, which are;

- The Ordinary Differential Equation
- The Partial Differential Equation
- The Linear Differential Equation
- The Non-Linear Differential Equation
- Homogeneous Differential Equation
- Non-Homogeneous Differential Equation

But prior to this research we shall be focusing on the Ordinary Differential Equations.

1.3. History of ordinary differential equation

The attempt to solve physical problems led gradually to mathematical models involving an equation in which a function and its derivatives play important roles. However, the theoretical development of this new branch of mathematics – Ordinary Differential Equations – has its origins rooted in a small number of mathematical problems. These problems and their solutions led to an independent discipline with the solution of such equations an end in itself. In circa 1671, English physicist Isaac Newton wrote his then-unpublished *The Method of Fluxions and Infinite Series* (which was later published in 1736), in which he classified first order differential equations, known to him as fluxional equation, into three classes:

$$\frac{dy}{dx} = f(x) \dots\dots\dots\text{CLASS 1}$$

$$\frac{dy}{dx} = f(x, y) \quad \dots\dots\dots\text{CLASS 2}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u \quad \dots\dots\dots\text{CLASS 3}$$

Where Classes 1 and 2 contains only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, which are currently known as ordinary differential equations; the third class involves the partial derivatives of one dependent variable that we call partial differential equations this day.

In 1676, Newton solved his first differential equation. That same year, Leibniz introduced the term “differential equation” (aequatio differentialis, which is a Latin word) or to denote a relationship between the differentials *dx and dy* of two variables x and y.

In 1693, Leibniz solved his first differential equation, and later that year Newton published the results of previous differential equation solution methods – a year that is said to mark the inception for the differential equations as a distant field in mathematics. Swiss mathematicians, Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748) they were known as the Bernoulli brothers, in Basel, Switzerland, were among the first interpreters of Leibniz’ version of differential calculus. They were both critical of Newton’s theories and maintained that Newton’s theory of fluxions was plagiarized from Leibniz’ original theories, and went to great lengths, using differential calculus, to disprove

Newton's *Principia*, on account that the brothers could not accept the theory, which Newton had proven, that the earth and the planets rotate around the sun in elliptical orbits. The first book on the subject of differential equations, supposedly, was an Italian Mathematician Gabriele Manfredi's 1707 *On the Construction of First Degree Differential Equations*, the book was written between 1701 and 1704, published in Latin. The book was solely based on the views of the Leibniz and Bernoulli brothers. Most of the publications on differential equations and partial differential equations, in the years to follow, in the 18th century, seemed to expand on the version developed by Leibniz, a methodology, employed by those as Leonhard Euler, Daniel Bernoulli, Joseph Lagrange and Pierre Laplace. In 1739, Swiss mathematician Leonhard Euler began using the integrating factor as an aid to derive differential equations that were integrable in finite form.

1.3.1. Applications of ordinary differential equations

ODEs has surprising applications and it has the ability to foresee our general surroundings. It is used in different disciplines like science, financial aspects, physical science, science and designing. It assists with foreseeing the outstanding development and rot, populace and species development. A portion of the purposes of ODEs are:

- Modelling the growth of diseases
- Describes the movement of electricity
- Describes the motion of the pendulum, waves
- Used in Newton's second law of motion and Law of cooling

1.4. Some basic definition on differential equation

- An equation including a connection between an unknown function and it's derivative(s) is known as a Differential Equation (DE). On the off chance that the independent variable in the equation is one, the differential equation (DE) is called an Ordinary Differential Equation (ODE), for instance, $y' = x^2$ where $y = f(x)$ is an ODE where y' represents $\frac{dy}{dx}$.
- A differential equation is an equation that uses derivatives or differentials to relate two or more variables. thus, $\frac{dy}{dx} = h(x)$, where $h(x)$ is a known function of the independent variable x , can be used to express a differential equation in its simplest form.
- In place of the continuous interval $[a, b]$ of x , discrete points $\{x_n\}$ defined by
- $x_n = a + nh, n = 0, 1, 2, \dots, N = \frac{(b-a)}{h}$
- The parameter h is called the step length.
- A computation error known as a truncation error happens when a value is not as accurate as it should be and results in an imprecise or wrong final response. Truncated data values might still be correct but they might not be precise. It is denoted by $O(h^{p+1})$.

1.5. Statement of the problem

Numerous fields of science, engineering, and the economics regularly use Ordinary Differential Equation (ODE) as mathematically models. Unfortunately, closed-form solutions to the equations are rarely found, so approximate solutions are frequently sought using numerical methods. These days, this can typically be done very cheaply, with high accuracy, and with a reliable bound on the error between the analytical solution and its numerical approximation. A first order differential equation is of the form:

$$y' = f(x, y) \dots\dots\dots(1.4.1)$$

for the real-valued function y of the real variable x ,

$$\text{where } y' = \frac{dy}{dx}$$

the differential equation will be considered together with an initial condition: given two real numbers x_0 and y_0 , we find a solution to (1.4.1)

for $x > x_0$ such that

$$y(x_0) = y_0 \dots\dots\dots(1.4.2)$$

The differential equation (1.4.1) together with the initial condition (1.4.2) is called an initial value problem.

1.6. The initial value problem (IVP)

An initial value problem (IVP) is an ordinary differential equation with an underlying condition (also known as the initial condition (1.4.2)) which determines the value of the unknown function at a given point in the space (or domain). An initial value problem (IVP) is an ODE and an initial condition that describes the value of the unknown function at a certain location or a given point in the domain. The general non-linear initial value problem;

$$y' = f(x, y), f: \mathcal{R} \times \mathcal{R}^m \rightarrow \mathcal{R}^m, y(a) = \eta \dots\dots\dots(1.5.1)$$

An example of an IVP is: $\frac{dy}{dx} = 10 - x, y(0) = -1$

Solution:

$$\frac{dy}{dx} = 10 - x \rightarrow dy = (10 - x)dx$$

$$\int dy = 10x - \frac{x^2}{2} + c, \text{ when } x = 0, y = -1$$

$$-1 = 10(0) - \frac{0}{2} + c \therefore c = -1$$

$$y = 10x - \frac{x^2}{2} - 1$$

1.7. Existence and uniqueness of solutions

We consider the first order differential equation of the form

$$\frac{dy}{dx} = f(x, y) \dots\dots\dots(1.6.1)$$

with initial condition;

$$y(x_0) = y_0 \dots\dots\dots(1.6.2)$$

The theorem below is to prove the existence and uniqueness of a solution.

1.8. Theorem of Existence and Uniqueness

Suppose $f(x, y)$ is a continuous function in a rectangle of the form $\{(x, y) | a < x < b, c < y < d\}$ in the xy -plane. If (x_0, y_0) is a point in this rectangle, then there exists an $\varepsilon > 0$ and a function $y(x)$ defined for

$$x_0 - \varepsilon < x < x_0 + \varepsilon$$

that solves the IVP (1.6.1) with initial condition (1.6.2).

Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous functions in a rectangle of the form

$$\{(x, y) | a < x < b, c < y < d\}$$

in the xy -plane. If (x_0, y_0) is a point in this rectangle, then there is a unique solution $y(x)$, defined on an interval

$$x_0 - \varepsilon < x < x_0 + e_0$$

for some $\varepsilon > 0$.

1.9. Aims and objectives of the study

The main aim of this project is to generate the Adams Bashforth method using the first order ordinary differential equation. Other specific objectives of this project are:

- Giving brief description on the One Step Euler method and the Runge Kutta method,
- To introduce the families of the Linear multistep method,
- The advantages of the Adams Bashforth method over the 4th Order Runge kutta method, and
- The major disadvantage of the Linear multistep method.

1.10. The Adams-Bashforth method

The Adams-Bashforth method is a numerical technique used for solving ordinary differential equations (ODEs), particularly initial value problems (IVPs). This method belongs to a class of explicit methods for numerical integration of ODEs and is particularly useful when the ODEs cannot be solved analytically or when analytic solutions are difficult to obtain.

1.11. Real-life applications of the Adams-Bashforth Method

- a. **Engineering Simulations:** In engineering, many systems are modeled using differential equations, such as those governing the behavior of electrical circuits, mechanical systems, chemical reactions, and fluid dynamics. The Adams-Bashforth method can be used to numerically simulate the behavior of these systems over time.
- b. **Climate Modeling:** Climate models simulate the Earth's climate system, which involves a complex set of differential equations representing interactions between the atmosphere, oceans, land surface, and ice. The Adams-Bashforth method can be employed to numerically solve these equations and predict climate patterns and changes over time.
- c. **Aerospace and Flight Dynamics:** Aerospace engineers use differential equations to model the motion of aircraft and spacecraft. The Adams-Bashforth method can help in numerically simulating the trajectory of a spacecraft or an aircraft under various conditions, aiding in design and analysis.
- d. **Biological Systems:** Differential equations are frequently used to model biological systems such as population dynamics, biochemical reactions, and physiological processes. The Adams-Bashforth method can be applied to simulate the behavior of these systems over time, helping researchers understand biological phenomena and develop treatments for diseases.
- e. **Economic Modeling:** Economic models often involve differential equations to describe the dynamics of markets, supply and demand, and macroeconomic variables. The Adams-Bashforth method can be used to numerically solve these equations and analyze the behavior of economic systems over time.
- f. **Computer Graphics and Animation:** Differential equations are used in computer graphics and animation to model physical phenomena like fluid flow, particle motion, and deformation of objects. The Adams-Bashforth method can be applied to simulate these phenomena and create realistic animations in video games, movies, and virtual reality environments.
- g. **Automotive Engineering:** Automotive engineers use differential equations to model vehicle dynamics, engine performance, and control systems. The Adams-Bashforth method is applied for simulating vehicle behavior under different driving conditions and optimizing performance.
- h. **Energy Sector:** The energy industry employs differential equations to model the behavior of power generation systems, including nuclear reactors, wind turbines, and electrical grids. The Adams-Bashforth method is used for simulating the transient response of these systems and optimizing their operation.
- i. **Environmental Engineering:** Environmental engineers use differential equations to model the dispersion of pollutants, groundwater flow, and ecological systems. The Adams-Bashforth method helps in simulating environmental processes and assessing the impact of human activities on the environment.
- j. **Chemical and Process Engineering:** Chemical engineers use differential equations to model chemical reactions, heat transfer, and fluid flow in industrial processes. The Adams-Bashforth method is applied for simulating process dynamics, optimizing production, and designing control strategies.
- k. **Biomedical Engineering:** Biomedical engineers use differential equations to model physiological systems, drug kinetics, and medical imaging techniques. The Adams-Bashforth method is used for simulating biological processes, designing medical devices, and developing treatment strategies.
- l. **Financial Services:** Financial analysts use differential equations to model financial markets, interest rates, and investment portfolios. The Adams-Bashforth method is applied for simulating market dynamics, pricing derivatives, and managing risk in financial institutions.

- m. **Telecommunications:** Telecommunication engineers use differential equations to model signal propagation, network traffic, and communication protocols. The Adams-Bashforth method is applied for simulating network behavior, optimizing performance, and designing protocols for wireless and wired communication systems.

1.12. Shortcomings of the Adams-Bashforth Method

While the Adams-Bashforth method is a widely used numerical technique for solving ordinary differential equations (ODEs), it does have some shortcomings:

- a. **Stability:** The Adams-Bashforth method can suffer from stability issues, particularly for stiff ODEs or when the step size is too large. If the step size is not chosen appropriately, the method may produce inaccurate or oscillatory solutions, leading to numerical instability.
- b. **Accuracy:** Although the Adams-Bashforth method is known for its simplicity and efficiency, it may not provide accurate solutions for certain types of ODEs, especially if the function being integrated is highly nonlinear or contains rapid changes. Higher-order Adams-Bashforth methods can improve accuracy but may introduce additional computational complexity.
- c. **Initial Condition Sensitivity:** The accuracy of the Adams-Bashforth method can be sensitive to the initial conditions. Small errors in the initial values or in the numerical integration process can propagate over time, leading to significant deviations from the true solution.
- d. **Limited to Explicit Formulation:** The Adams-Bashforth method is an explicit method, meaning that the solution at each time step depends only on previous time steps. While this makes the method computationally efficient, it also limits its applicability to certain types of ODEs. Implicit methods, which consider future time steps in addition to past ones, may be more stable for stiff equations.
- e. **Inability to Handle Adaptive Step Sizes:** Unlike some other numerical methods, such as the Runge-Kutta methods, the Adams-Bashforth method does not naturally support adaptive step size control. Adaptive step size algorithms are useful for efficiently integrating ODEs with varying levels of stiffness or rapidly changing dynamics.
- f. **Limited Order of Accuracy:** The order of accuracy of the Adams-Bashforth method is limited by its construction. While higher-order Adams-Bashforth methods exist, achieving higher orders of accuracy can be challenging due to the explicit nature of the method and the need to compute coefficients for the method.

2. Literature review

2.1. Introduction

An approximate computer method known as a Numerical Method is used to solve mathematical problems, many of which lack an analytical solution. We really want estimation since we either can't settle the system systematically or in light of the fact that the scientific technique is immovable. We shall briefly discuss the Existing Method and the Linear Multistep Method.

2.2. The existing method

There are two existing methods we will be discussing, which are;

- The One Step method and
- The Linear Multistep Method

2.3. The One Step Method

The One Step method general formula is written as:

$$y_{n+1} = y_n + h_n \Phi(x_n, y_n; h_n) \dots\dots\dots(2.2.1)$$

where $\Phi(x_n, y_n; h_n)$ is the increment function and h_n is the step length adopted in a sub-interval $\{x_n, x_{n+1}\}$. For the sake of convenience and easy computation, h_n will be fixed i.e. $h_n = h, n \geq 0$. The One Step method is divided into two methods namely;

- The Euler Method and
- The Runge Kutta Method

2.3.1. The Euler Method

Euler’s method is the most straightforward one stage method with increment function $\Phi(x, y; h)$ specified as

$$\Phi(x, y; h) = f(x, y) \dots\dots\dots(2.2.2)$$

so that (2.2.1) becomes

$$y_{n+1} = y_n + hf(x_n, y_n) \dots\dots\dots(2.2.3)$$

Moreover, the Euler scheme (2.2.3) can be derived easily using the Taylor expansion of $y(x_{n+1})$ about $x = x_n$. It is the fundamental explicit method for mathematical integration for Ordinary Differential Equations. Euler proposed his technique for initial value problem in 1768. It is for solving initial value problem (IVP), and it is a productive mathematical (numerical) method however not exact or accurate, yet a significant number of the thoughts associated with the mathematical (numerical) solution of differential equations are presented most basically with it.

Although, the Euler’s method is only first order accurate, it is straightforward and simple to apply because it only requires one evaluation of the function. The one disadvantage with Euler method is that it has a lower order of $p = 1$ which then brought about the Runge Kutta Method.

2.3.2. The Runge Kutta Method

In a mathematical examination (numerical analysis), the Runge Kutta methods are a group of explicit and implicit iterative method. These methods were created around 1900 by the German Mathematicians Carl Runge and Wilhelm Kutta. The general formula for Runge Kutta method is:

$$y_{n+1} = y_n + h\Phi_{RK}(x_n, y_n, h) \dots\dots\dots(2.2.4)$$

with the increment function Φ_{RK} given as a weighted mean of the slopes at specific points $\{Z_r | Z_r = x_n + c_r, 0 \leq c_r \leq 1\}$.

$$\Phi_{RK} = \sum_{j=1}^s b_j Y_j, s \geq 1 \dots\dots\dots(2.2.5)$$

with the constraint

$$\sum_{j=1}^s b_j = 1 \dots\dots\dots(2.2.6)$$

Here, s is the number of stages of the process, and condition (2.2.6) makes sure that the integration scheme is consistent.

The Runge Kutta method can be thought as an extension of the Taylor series expansion scheme, with the evaluation of the first and higher order derivative of $f(x, y)$ is being replaced by an evaluation of a function inside each interval of integration. The RKM being categorized into the three classes are;

- The explicit Runge Kutta if $a_{rj} = 0$, for all $j \geq r$,
- The semi implicit Runge Kutta if $a_{rj} = 0$ for all $j > r$, and also,
- The implicit Runge Kutta if $a_{rj} \neq 0$, for at least one $j > r$

while the orders of the Runge Kutta method are as follow:

- Runge Kutta of order 1
- Runge Kutta of order 2
- Runge Kutta of order 3
- Runge Kutta of order 4

The members of the explicit Runge-Kutta method is a generalization of the RK4 method and it’s given as;

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i k_i \dots\dots\dots(2.2.5)$$

where,

$$\begin{aligned}
 k_1 &= f(x_n, y_n) \\
 k_2 &= f(x_n + c_2h, y_n + h(a_{21}k_1)), \\
 k_3 &= f(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))
 \end{aligned}$$

$$k_1 = f(x_n + c_s h, y_n + h(a_{s1}k_1 + a_{s2}k_2 + \dots))$$

to determine a specific method, one necessities to give the whole number s (the number of stages), and the coefficients a_{ij} (for $1 \leq j < i \leq s$), b_i (for $i = 1, 2, \dots, s$) and c_i (for $i = 2, 3, \dots, s$). The matrix $[a_{ij}]$ is called the Runge-Kutta matrix, while the b_i and c_i are known as the loads and nodes. These information are normally organized in a memory helper, known as the *Butcher Tableau*

Table 1 The Butcher’s Tableaux

C	A	=	0					
0	b^T		c_2	a_{21}				
		c_3	a_{31}	a_{32}				
		.	.	.				
				
		c_s	a_{s1}	a_{s2}	...	$a_{s,s-1}$		
			b_1	b_2	...	$b_{s,s-1}$	b_s	

The Runge Kutta Method take some intermediate step to get a higher order method and then discard all previous information before taking a second step. This is a major disadvantage which led to the Linear Multistep method.

2.4. The linear multistep method

The Linear Multistep Method (LMM) are used for mathematical solution of Ordinary Differential Equation (ODE). Thoughtfully, a mathematical method begins from an initial point and afterward **moves** forward so as to find the following solution points. The Linear Multistep Method formula is stated below:

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} \dots \dots \dots (2.3.1)$$

where,

$$|\alpha_0| + |\beta_0| \neq 0 \text{ and } \alpha_k \neq 0, \quad \alpha_k = 1 \dots \dots \dots (2.3.2)$$

2.4.1. Properties of linear multispep method

- **Definition of Absolute Stability:** the linear multistep method (2.3.1) is said to be absolutely stable for a given Z , if for that Z , all the roots of the stability polynomial satisfies $|r_t| < 1, t = 1, 2, \dots, k$ and absolutely unstable for that Z otherwise.
- **Definition of Region of Absolute Stability:** the LMM is said to have a region of absolute stability R_A , where R_A , is a region of complex - Z -plane, if it is absolute stable for all Z in R_A , (i.e. $Z \in R_A$) the intersection of R_A with the real $R_e(Z)$ axis is called the region of absolute stability.
- **Definition of A-stability:** a numerical method for the integration of IVP, is said to be A-stable, if its region of absolute R_A contains the entire left of the complex plane.
- **Definition of Zero Stability:** a LMM is said to be zero stable if it satisfies the root condition i.e. if $|r_j| \leq 1$.
- **Definition of Consistency:** the LMM is said to be consistent if the increment function $\varphi(x, y, 0) = f(x, y)$.

- **Definition of Convergence:** it is said to be convergent if for any arbitrary initial solution method y_0 and an arbitrary point $x \in [a, b]$, the global error satisfies

$$\lim_{n \rightarrow 0} |e_n| \rightarrow 0$$

The most commonly used Linear Multistep Method is the Backward Differentiation Formulas (BDF)

2.4.2. The backward differentiation formulas (BDF)

The BDF methods are implicit methods with $b_{k-1} = \dots = b_0$ and the different coefficients picked to such an extent that the method achieves order k (the greatest possible). These methods are mainly used for the solution of stiff/solid differential equations. The BDF general formula is given as:

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \beta_k f_{n+k}, \beta_k = 1 \dots\dots\dots(2.3.4)$$

The BDF is $A - stable$ for $k = 1$ and $k = 2$ and $A(\alpha) - stable$ for $k = 3$ through $k = 6$ but it is unstable for $k = 7$ and beyond, below is the BDF plots for $k=1$ to $k=7$;

For $k=1$ the plot is given as;

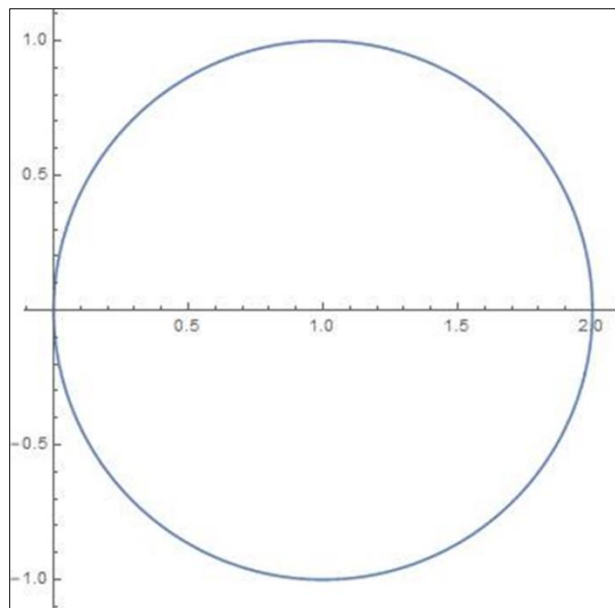


Figure 1 The Stability Plot of BDF for $k=1$

For $k=2$ the plot is given as;

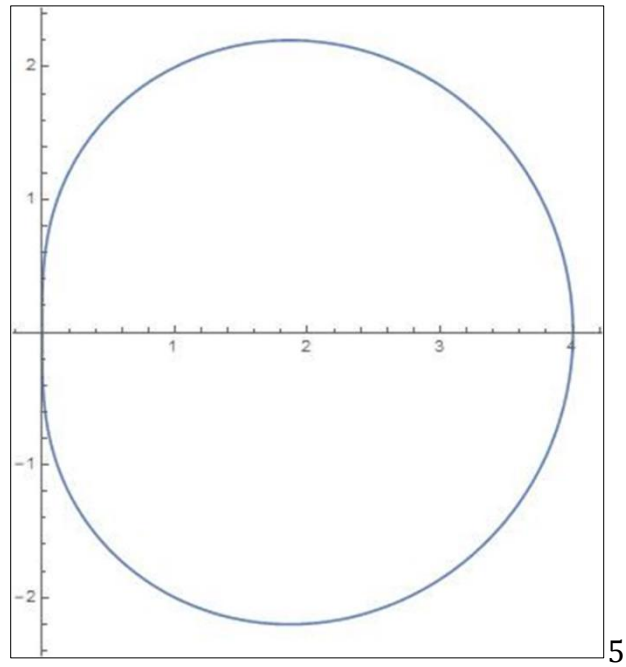


Figure 2 The Stability Plot of BDF for $k=2$

For the plot $k=3$, we have;

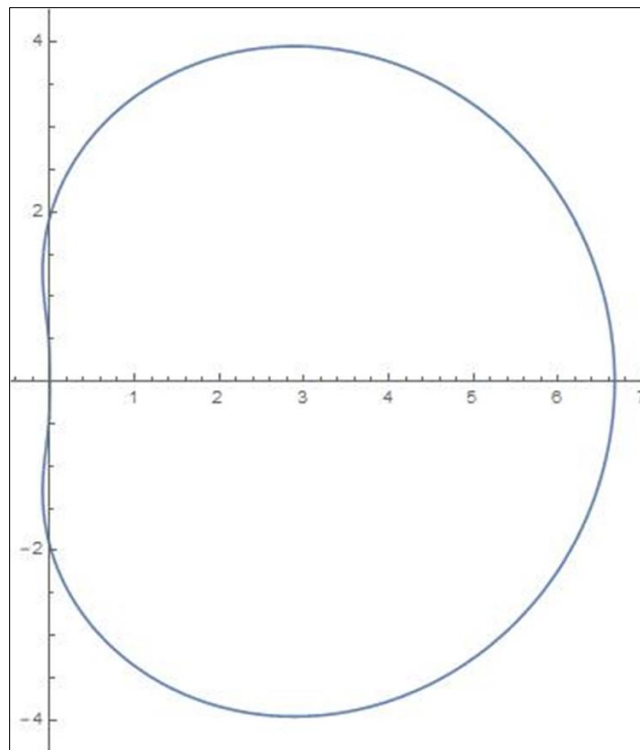


Figure 3 The Stability Plot for $k=3$

To plot for $k=4$, we have;

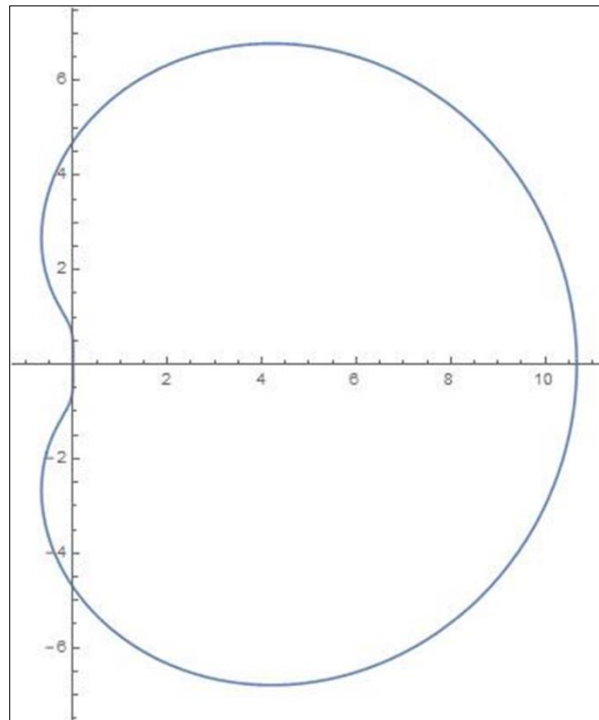


Figure 4 The Stability Plot of BDF for k=4

Plotting for k=5, we are to have,

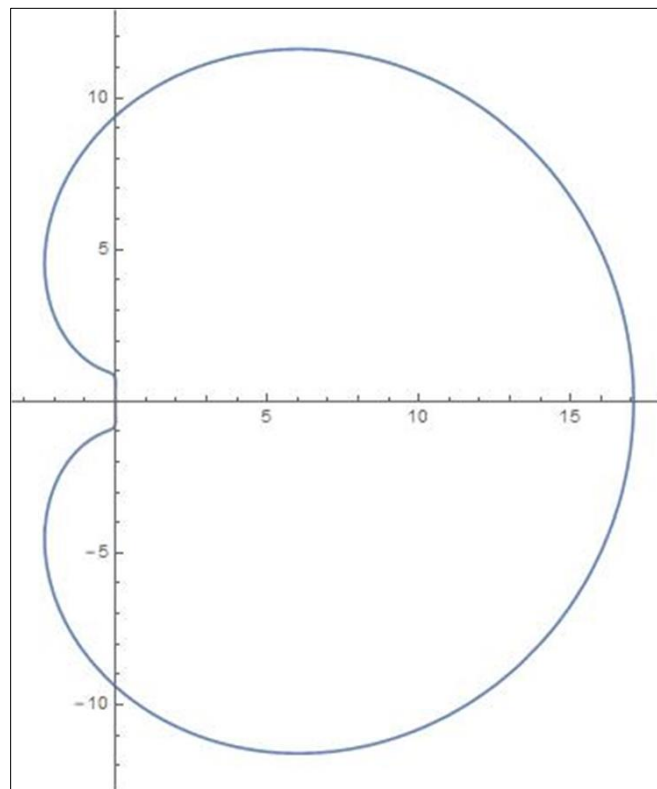


Figure 5 The Stability Plot of BDF for k=5

For k=6 the plot is;

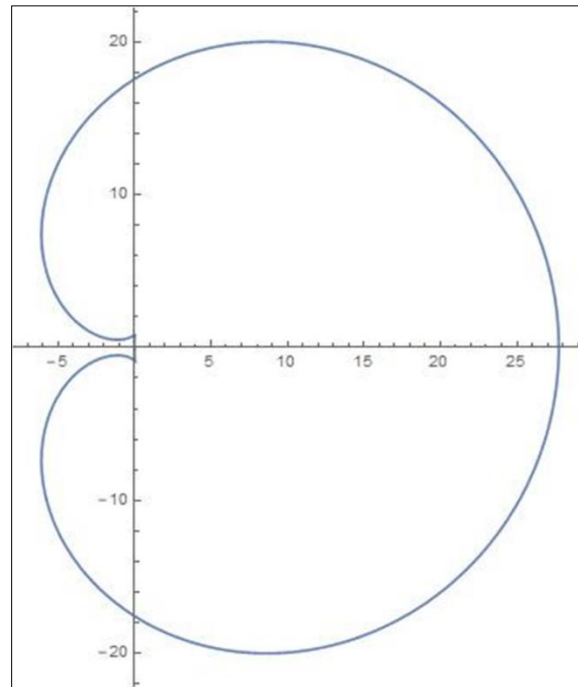


Figure 6 The Stability Plot of BDF for k=6

Plotting for k=7, we are to have,

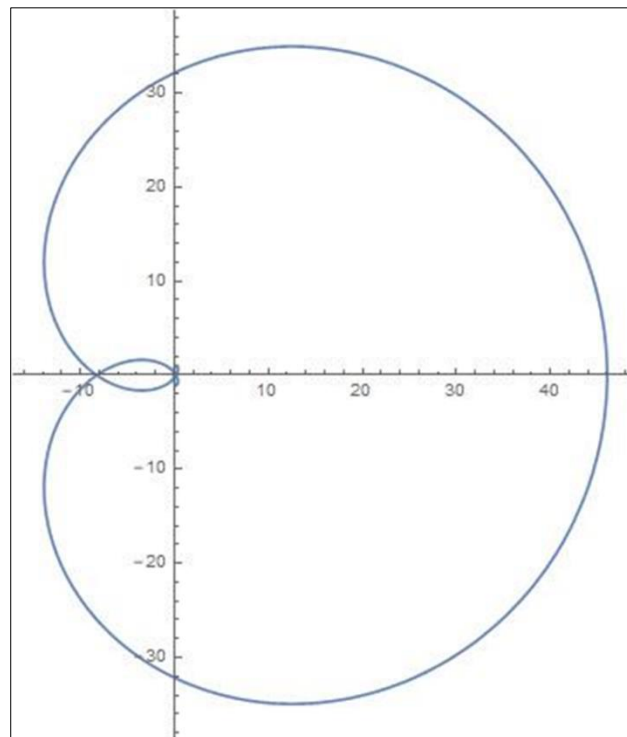


Figure 7 The Stability Plot of BDF for k=7

Other classes of the Linear Multistep Method are;

2.4.3. The adams moulton method

The Adams-Moulton methods are implicit methods in that they have $a_{k-1} = -1$ and $a_{k-2} = \dots = a_0 = 0$. And also, the b coefficients are preferred to get in the highest order possible. The Adams Moulton formula is:

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} = h \sum_{j=0}^k \beta_j f_{n+j} \dots\dots\dots(2.3.5)$$

Nevertheless, the Adams-Moulton method being an implicit method, by removing the restriction that $b_k = 0$, a k-step Adams Moulton get to order $k + 1$.

The Adams Moulton methods with $k = 0, 1, 2, 3, 4$ are (Hairer et al 1993)

$$y_n = y_{n-1} + hf(x_n, y_n) \quad \text{its known as the backward Euler method, and}$$

$$y_{n+1} = y_n + \frac{1}{2}h(f(x_{n+1}, y_{n+1}) + f(x_n + y_n)), \quad \text{known as the trapezoidal rule}$$

2.4.4. *The adams bashforth method*

The Adams Bashforth method is an explicit linear multistep method.

3. Methodology

3.1. Introduction

3.1.1. *The adams bashforth method*

The Adams Bashforth method were designed by John Couch Adams to solve a differential equation modelling capillary action due to Francis Bashforth in 1883 published his theory and Adams' numerical method (Goldstine 1997). The Adams Bashforth method is given as:

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} = h \sum_{j=0}^{k-1} \beta_j f_{n+j} \quad \dots\dots\dots(3.0)$$

The Adams Bashforth method is an explicit method with coefficients $a_{k-1} = -1$ and $a_{k-2} = \dots = a_0 = 0$, while the β_j are selected so that the procedures have an order of k (this determines the methods uniquely).

3.2. Derivation of the Adams Bashforth method

In this section we shall be deriving the Adams Bashforth method using the Taylor's series expansion with a Software Application called **Mathematica**.

3.2.1. *Deriving the Adams Bashforth method/scheme*

$$y_{n+k} - \alpha_0 y_{n+k-1} = h \sum_{j=0}^{k-1} \beta_j f_{n+j} \quad \dots\dots\dots(3.1)$$

for $k=1$ in (3.1) gives,

$$y_{n+1} - \alpha_0 y_n = h \sum_{j=0}^{1-1} \beta_j f_{n+j}$$

$$y_{n+1} - \alpha_0 y_n = h(\beta_0 f_n) \dots\dots\dots(3.2)$$

By expanding (3.2) using the Taylor's series results into the following

$$y_{n+1} = y[x + h] = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \dots \dots\dots(3.3)$$

$$\alpha_0 y_n = y(x) = \alpha_0 y(x) \quad \dots\dots\dots(3.4)$$

$$h\beta_0 f_n = y'(x) = h\beta_0 \dots\dots\dots(3.5)$$

By substituting (3.2), (3.3), and (3.4) into (3.1) we have,

$$y(x) + hy'(x) + \frac{h^2}{2!} y''(x) - \alpha_0 y(x) - h\beta_0 y'(x) = 0 \quad \dots\dots\dots(3.6)$$

we obtained the coefficients of (3.6) in terms of power of h;

$$\begin{aligned} h^0 &= 1 - \alpha_0 = 0 & \alpha_0 &= 1 \\ h^1 &= 1 - \beta_0 = 0 & \beta_0 &= 1 \\ h^2 &= \frac{1}{2} \neq 0 \end{aligned}$$

The coefficient $\alpha_0=1$ and $\beta_0=1$ is substituted into (3.2) to give

$$y_{n+1} - y_n = hf_n$$

For k=2

$$y_{n+2} - \alpha_0 y_{n+1} = h \sum_{j=0}^1 \beta_j f_{n+j}$$

$$y_{n+2} - \alpha_0 y_{n+1} = h(\beta_0 f_n + \beta_1 f_{n+1}) \dots\dots\dots(3.7)$$

The coefficients β_0 and β_1 is determined through the use of Taylor's series expansion of (3.7)

$$y_{n+2} = y(x + 2h) = y(x) + 2hy'(x) + \frac{(2h)^2}{2!} y''(x) + \frac{(2h)^3}{3!} y'''(x) \dots\dots\dots(3.8)$$

$$y_{n+1} = y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) \dots\dots\dots(3.9)$$

$$\alpha_0 y_{n+1} = y(x) = \alpha_0 y(x) + \alpha_0 h y'(x) + \alpha_0 \frac{h^2}{2!} y''(x) \dots\dots\dots(3.10)$$

$$h\beta_0 f_n = y'(x) = h\beta_0 y'(x) \dots\dots\dots(3.11)$$

$$h\beta_1 f_{n+1} = y'(x + h) = h\beta_1 y'(x) + h^2 \beta_1 y''(x) + \frac{h^3}{2!} y'''(x) + \frac{h^4}{3!} y^{(iv)}(x) \dots\dots\dots(3.12)$$

from equations (3.7-3.11) we collect the power of h

$$h^0 = 1 - \alpha_0 \qquad \alpha_0 = 1$$

$$h^1 = 2 - 1 - \beta_0 - \beta_1 = 0 \Rightarrow 1 - \beta_0 - \beta_1 = 0$$

$$1 - \beta_0 - \frac{3}{2}$$

$$-\frac{1}{2} - \beta_0 = 0, \beta_0 = -\frac{1}{2}$$

$$h^2 = 2 - \frac{1}{2} - \beta_1 = 0 \qquad \beta_1 = \frac{3}{2}$$

$$\therefore \alpha_0 = 1, \beta_0 = -\frac{1}{2}, \beta_1 = \frac{3}{2}$$

The coefficients $\alpha_0 = 1, \beta_0 = -\frac{1}{2}, \beta_1 = \frac{3}{2}$ is substituted into (3.7) to give

$$y_{n+2} - y_{n+1} = h\left(-\frac{1}{2}f_n + \frac{3}{2}f_{n+1}\right)$$

For k=3

$$y_{n+3} - \alpha_0 y_{n+2} = h \sum_{j=0}^2 \beta_j f_{n+j}$$

$$y_{n+3} - \alpha_0 y_{n+2} = h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}) \dots\dots\dots(3.13)$$

The coefficients β_0, β_1 and β_2 is determined through the use of Taylor's series expansion of (3.13) we have

$$y_{n+3} = y(x + 3h) = y(x) + 3hy'(x) + \frac{(3h)^2}{2!} y''(x) + \frac{(3h)^3}{3!} y'''(x) + \frac{(3h)^4}{4!} y^{iv}(x) \dots\dots\dots (3.14)$$

$$y_{n+2} = y(x + 2h) = y(x) + 2hy'(x) + \frac{(2h)^2}{2!} y''(x) + \frac{(2h)^3}{3!} y'''(x) + \frac{(2h)^4}{4!} y^{iv}(x) \dots\dots (3.15)$$

$$y_{n+1} = y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \frac{h^4}{4!} y^{iv}(x) \dots\dots\dots (3.16)$$

$$\alpha_0 y_{n+2} = \alpha_0 y(x + 2h) = \alpha_0 y(x) + \alpha_0 2hy'(x) + \alpha_0 \frac{(2h)^2}{2!} y''(x) + \alpha_0 \frac{(2h)^3}{3!} y'''(x) + \alpha_0 \frac{(2h)^4}{4!} y^{iv}(x) \dots\dots\dots (3.17)$$

$$h\beta_0 f_n = y'(x) = h\beta_0 y'(x) \dots\dots\dots(3.18)$$

$$h\beta_1 f_{n+1} = y'(x + h) = h\beta_1 y'(x) + h^2 \beta_1 y''(x) + \frac{h^3}{2!} \beta_1 y'''(x) + \frac{h^4}{3!} \beta_1 y^{iv}(x) + \frac{h^5}{4!} \beta_1 y^v(x) \dots\dots\dots(3.19)$$

$$h\beta_2 + f_{n+2} = y'(x + 2h) = 2h\beta_2 y'(x) + (2h)^2 \beta_2 y''$$

from equations (3.14-3.19) we have for power of h

$$h^0 = 1 - \alpha_0 = 0 \quad \alpha_0 = 1$$

$$h^1 = 3 - 2\alpha_0 - \beta_0 - \beta_1 - \beta_2 = 0$$

$$h^2 = \frac{9}{2} - 2\alpha_0 - \beta_1 - 2\beta_2 = 0$$

$$h^3 = \frac{9}{2} - \frac{4}{3}\alpha_0 - \frac{1}{2}\beta_1 - \frac{2}{3}\beta_2$$

The coefficients $\alpha_0 = 1, \beta_0 = \frac{5}{12}, \beta_1 = -\frac{4}{3}$ and $\beta_2 = \frac{23}{12}$ is substituted to give:

$$y_{n+3} - y_{n+2} = h(\frac{5}{12} f_n - \frac{4}{3} f_{n+1} + \frac{23}{12} f_{n+2})$$

For k=4

$$y_{n+4} - \alpha_0 y_{n+3} = h \sum_{j=0}^3 \beta_j f_{n+j}$$

$$y_{n+4} - \alpha_0 y_{n+3} = h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3}) \dots\dots\dots(3.20)$$

The coefficients $\beta_0, \beta_1, \beta_2$ and β_3 is determined through the use of Taylor's series expansion of (3.20), we have

$$y_{n+4} = y(x + 4h) = y(x) + 4hy'(x) + \frac{(4h)^2}{2!} y''(x) + \frac{(4h)^3}{3!} y'''(x) + \frac{(4h)^4}{4!} y^{iv}(x) + \frac{(4h)^5}{5!} y^v(x) \dots\dots\dots(3.21)$$

$$y_{n+3} = y(x + 3h) = y(x) + 3hy'(x) + \frac{(3h)^2}{2!} y''(x) + \frac{(3h)^3}{3!} y'''(x) + \frac{(3h)^4}{4!} y^{iv}(x) + \frac{(3h)^5}{5!} y^v(x) \dots\dots\dots(3.22)$$

$$y_{n+2} = y(x + 2h) = y(x) + 2hy'(x) + \frac{(2h)^2}{2!}y''(x) + \frac{(2h)^3}{3!}y'''(x) + \frac{(2h)^4}{4!}y^{iv}(x) + \frac{(2h)^5}{5!}y^v(x) \dots\dots\dots(3.23)$$

$$y_{n+1} = y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{iv}(x) + \frac{h^5}{5!}y^v(x) \dots\dots\dots(3.24)$$

$$\alpha_0 y_{n+3} = \alpha_0 y(x + 3h) = \alpha_0 y(x) + \alpha_0 3hy'(x) + \alpha_0 \frac{(3h)^2}{2!}y''(x) + \alpha_0 \frac{(3h)^3}{3!}y'''(x) + \alpha_0 \frac{(3h)^4}{4!}y^{iv}(x) + \frac{(3h)^5}{5!}y^v(x) \dots(3.25)$$

$$h\beta_0 f_n = y'(x) = h\beta_0 y'(x) \dots\dots\dots(3.26)$$

$$h\beta_1 f_{n+1} = y'(x + h) = h\beta_1 y'(x) + h^2\beta_1 y''(x) + \frac{h^3}{2!}\beta_1 y'''(x) + \frac{h^4}{3!}\beta_1 y^{iv}(x) + \frac{h^5}{4!}\beta_1 y^v(x) \dots\dots\dots(3.27)$$

$$h\beta_2 f_{n+2} = y'(x + 2h) = 2h\beta_2 y'(x) + (2h)^2\beta_2 y''(x) + \frac{(2h)^3}{2!}\beta_2 y'''(x) + \frac{(2h)^4}{3!}\beta_2 y^{iv}(x) + \frac{(2h)^5}{4!}\beta_2 y^v(x) \dots\dots\dots(3.28)$$

$$h\beta_3 f_{n+3} = y'(x + 3h) = 3h\beta_3 y'(x) + (3h)^2\beta_3 y''(x) + \frac{(3h)^3}{2!}\beta_3 y'''(x) + \frac{(3h)^4}{3!}\beta_3 y^{iv}(x) + \frac{(3h)^5}{4!}\beta_3 y^v(x) \dots\dots\dots(3.29)$$

from equations (3.21-3.29) we have for power of h

$$h^0 = 1 + \alpha_0 = 0 \quad \therefore \alpha_0 = 1$$

$$h^1 = 4 - 3\alpha_0 - \beta_0 - \beta_1 - \beta_2 - \beta_3$$

$$h^2 = 8 - \frac{9}{2}\alpha_0 - \beta_1 - 2\beta_2 - 3\beta_3 = 0$$

$$h^3 = \frac{32}{6} - \frac{9}{2}\alpha_0 - \frac{1}{2}\beta_1 - 2\beta_2 - 4\beta_3 = 0$$

$$h^4 = \frac{32}{3} - \frac{27}{8}\alpha_0 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 = 0$$

The coefficients $\alpha_0 = 1, \beta_0 = -\frac{3}{8}, \beta_1 = \frac{37}{24}, \beta_2 = -\frac{59}{24}$ and $\beta_3 = \frac{55}{24}$ is substituted to give:

$$y_{n+4} - y_{n+3} = h\left(-\frac{3}{8}f_n + \frac{37}{24}f_{n+1} - \frac{59}{24}f_{n+2} + \frac{55}{24}f_{n+3}\right)$$

For k=5

$$y_{n+5} - \alpha_0 y_{n+4} = h \sum_{j=0}^4 \beta_j f_{n+j}$$

$$y_{n+5} - \alpha_0 y_{n+4} = h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4}) \dots\dots\dots(3.30)$$

The coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ and β_4 is determined through the use of Taylor's series expansion of (3.30);

$$y_{n+5} = y(x + 5h) = y(x) + 5hy'(x) + \frac{(5h)^2}{2!}y''(x) + \frac{(5h)^3}{3!}y'''(x) + \frac{(5h)^4}{4!}y^{iv}(x) + \frac{(5h)^5}{5!}y^v(x) + \frac{(5h)^6}{6!}y^{vi}(x) \dots\dots\dots(3.31)$$

$$y_{n+4} = y(x + 4h) = y(x) + 4hy'(x) + \frac{(4h)^2}{2!}y''(x) + \frac{(4h)^3}{3!}y'''(x) + \frac{(4h)^4}{4!}y^{iv}(x) + \frac{(4h)^5}{5!}y^v(x) + \frac{(4h)^6}{6!}y^{vi}(x) \dots\dots\dots(3.32)$$

$$y_{n+3} = y(x + 3h) = y(x) + 3hy'(x) + \frac{(3h)^2}{2!}y''(x) + \frac{(3h)^3}{3!}y'''(x) + \frac{(3h)^4}{4!}y^{iv}(x) + \frac{(3h)^5}{5!}y^v(x) + \frac{(3h)^6}{6!}y^{vi}(x) \dots\dots\dots(3.33)$$

$$y_{n+2} = y(x + 2h) = y(x) + 2hy'(x) + \frac{(2h)^2}{2!}y''(x) + \frac{(2h)^3}{3!}y'''(x) + \frac{(2h)^4}{4!}y^{iv}(x) + \frac{(2h)^5}{5!}y^v(x) + \frac{(2h)^6}{6!}y^{vi}(x) \dots\dots\dots(3.34)$$

$$y_{n+1} = y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)}(x) + \frac{h^5}{5!}y^{(5)}(x) + \frac{h^6}{6!}y^{(6)}(x) \dots\dots\dots(3.35)$$

$$\alpha_0 y_{n+4} = \alpha_0 y(x + 4h) = \alpha_0 y(x) + \alpha_0 4hy'(x) + \alpha_0 \frac{(4h)^2}{2!}y''(x) + \alpha_0 \frac{(4h)^3}{3!}y'''(x) + \alpha_0 \frac{(4h)^4}{4!}y^{(4)}(x) + \alpha_0 \frac{(4h)^5}{5!}y^{(5)}(x) + \alpha_0 \frac{(4h)^6}{6!}y^{(6)}(x) \dots\dots\dots(3.36)$$

$$h\beta_0 f_n = y'(x) = h\beta_0 y'(x) \dots\dots\dots(3.37)$$

$$h\beta_1 f_{n+1} = y'(x + h) = h\beta_1 y'(x) + h^2\beta_1 y''(x) + \frac{h^3}{2!}\beta_1 y'''(x) + \frac{h^4}{3!}\beta_1 y^{(4)}(x) + \frac{h^5}{4!}\beta_1 y^{(5)}(x) + \frac{h^6}{5!}\beta_1 y^{(6)}(x) \dots\dots\dots(3.38)$$

$$h\beta_2 f_{n+2} = y'(x + 2h) = 2h\beta_2 y'(x) + (2h)^2\beta_2 y''(x) + \frac{(2h)^3}{2!}\beta_2 y'''(x) + \frac{(2h)^4}{3!}\beta_2 y^{(4)}(x) + \frac{(2h)^5}{4!}\beta_2 y^{(5)}(x) + \frac{(2h)^6}{5!}\beta_2 y^{(6)}(x) \dots\dots\dots(3.39)$$

$$h\beta_3 f_{n+3} = y'(x + 3h) = 3h\beta_3 y'(x) + (3h)^2\beta_3 y''(x) + \frac{(3h)^3}{2!}\beta_3 y'''(x) + \frac{(3h)^4}{3!}\beta_3 y^{(4)}(x) + \frac{(3h)^5}{4!}\beta_3 y^{(5)}(x) + \frac{(3h)^6}{5!}\beta_3 y^{(6)}(x) \dots\dots\dots(3.40)$$

$$h\beta_4 f_{n+4} = y'(x + 4h) = 4h\beta_4 y'(x) + (4h)^2\beta_4 y''(x) + \frac{(4h)^3}{2!}\beta_4 y'''(x) + \frac{(4h)^4}{3!}\beta_4 y^{(4)}(x) + \frac{(4h)^5}{4!}\beta_4 y^{(5)}(x) + \frac{(4h)^6}{5!}\beta_4 y^{(6)}(x) + \dots \dots\dots(3.41)$$

from equations (3.38-3.41) we collect for power of h

$$h^0 = 1 - \alpha_0 = 0$$

$$h^1 = 5 - 4\alpha_0 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 = 0$$

$$h^2 = \frac{25}{2} - 8\alpha_0 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4 = 0$$

$$h^3 = \frac{125}{6} - \frac{32}{3}\alpha_0 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 - 8\beta_4 = 0$$

$$h^4 = \frac{625}{24} - \frac{32}{3}\alpha_0 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 - \frac{32}{3}\beta_4 = 0$$

$$h^5 = \frac{625}{24} - \frac{128}{15}\alpha_0 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 - \frac{32}{3}\beta_4 = 0$$

The coefficients $\alpha_0 = 1, \beta_0 = \frac{251}{720}, \beta_1 = -\frac{637}{360}, \beta_2 = \frac{109}{30}, \beta_3 = -\frac{1387}{360}$ and $\beta_4 = \frac{1901}{720}$ is then substituted into (3.30)

$$y_{n+5} - y_{n+4} = h\left(\frac{251}{720}f_n - \frac{637}{360}f_{n+1} + \frac{109}{30}f_{n+2} - \frac{1387}{360}f_{n+3} + \frac{1901}{720}f_{n+4}\right)$$

3.3. Stability Analysis for Adams Bashforth

The application of Adams Bashforth to scalar test equation

$$y' = \lambda y, Re(\lambda) < 0$$

gives

$$y_{n+k} - \alpha_0 y_{n+k-1} = h\lambda \sum_{j=0}^{k-1} \beta_j y_{n+j}$$

Where $h\lambda = z$, the shift operator $r^j y_n = y_{n+j}$. Thus we have

$$\left(r^k - \alpha_0 r^{k-1} - z \sum_{j=0}^{k-1} \beta_j r^j \right) y_n = 0$$

The stability function for the Adams Bashforth is given as

$$r^k - \alpha_0 r^{k-1} - z \sum_{j=0}^{k-1} \beta_j r^j$$

The plot for the case of k=1(1)5 is given in Figs 3.1 - 3.5. The method is limited in stability region that is the stable region is the interior of the closed curve, see Table 3.1 for the interval stability region.

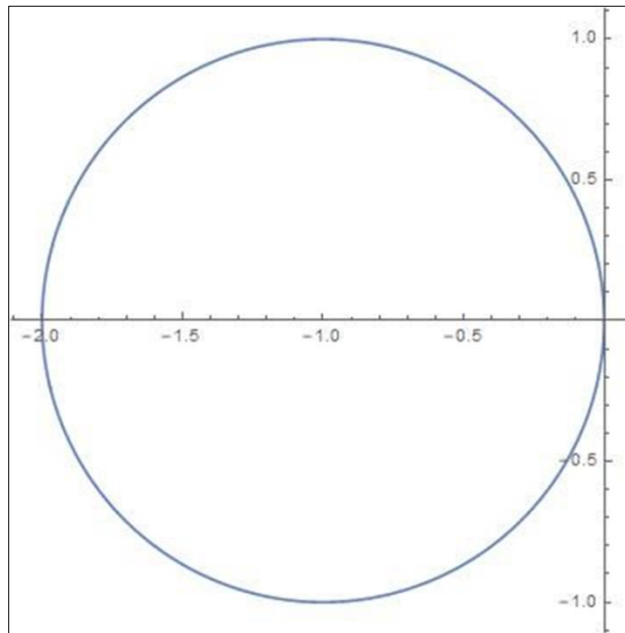


Figure 8 This stability plot for Adams Bashforth's Method for k=1

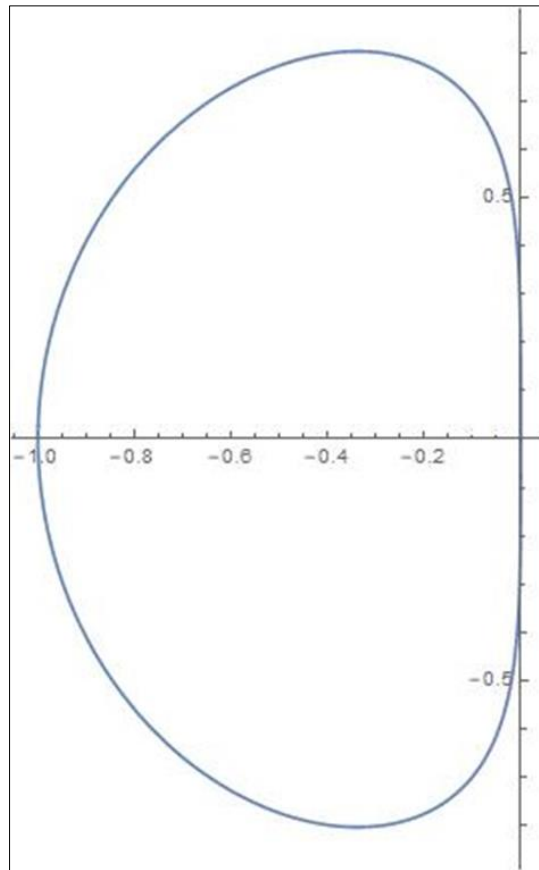


Figure 9 The stability plot for Adams Bashforth Method for $k=2$

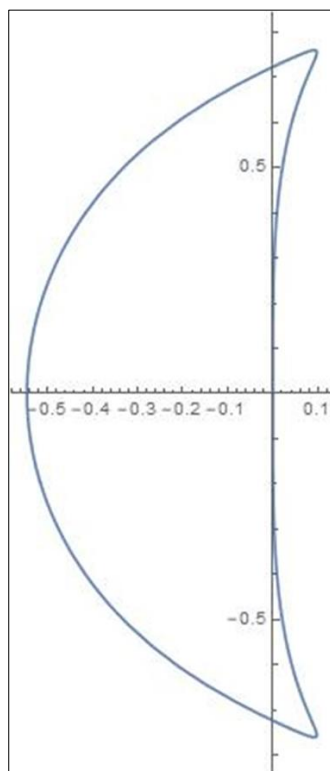


Figure 10 The Stability Plot for Adams Bashforth Method for $k=3$

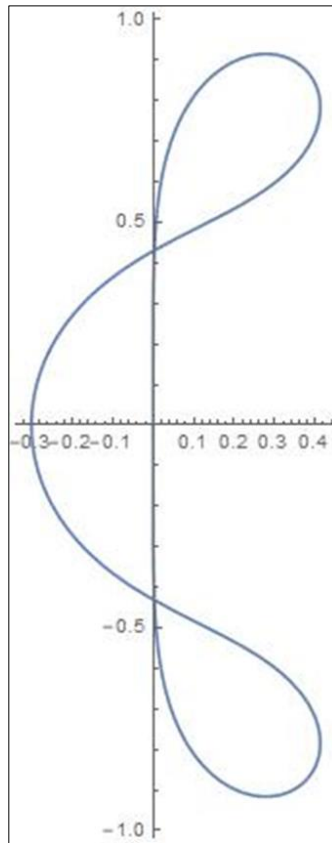


Figure 11 The Stability plot for Adams Bashforth Method for $k=4$

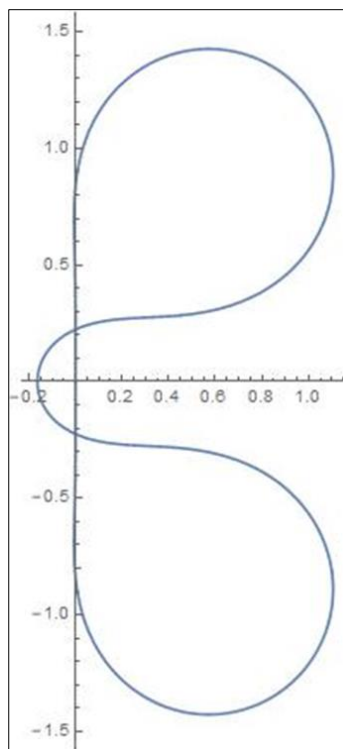


Figure 12 The Stability plot for Adams Bashforth Method for $k=5$

Table 2 The Stability Interval and Error Constant of the Adams Bashforth for $k = 1$ (1)5

<i>step lenght k</i>	<i>Region of Stability</i>	<i>Error Constant of Adams Bashforth</i>
1	$[-0.20, 0]$	$\frac{1}{2}$
2	$[-0.10, 0]$	$\frac{5}{12}$
3	$[-0.54, 0]$	$\frac{3}{8}$
4	$[-0.32, 0]$	$\frac{251}{720}$
5	$[-0.18, 0]$	$\frac{95}{288}$

3.4. Implementation of the method with example

In this section, we shall test the Adams Bashforth with a Linear problem of the form

$$y' = y - x, \quad x_0 = 0, y_0 = \frac{1}{2}$$

Where the exact value is $y(x) = x + 1 - \frac{1}{2}e^x$ and $h = 0.1$

The past values of Adams Bashforth method are obtained from the lower version of it.

Using the Adams Bashforth method to solve at $k=1$

$$y_{n+1} = y_n + h f_n \dots\dots\dots(4.1)$$

When $n=0$ in (4.1)

$$y_1 = y_0 + h f_0$$

$$f_0 = y_0 - x_0 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$y_1 = \frac{1}{2} + \left(0.1 \times \frac{1}{2}\right) = \frac{1}{2} + 0.05 = 0.55$$

When $n=1$ in (4.1)

$$y_2 = y_1 + h f_1$$

$$f_1 = y_1 - x_1$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$f_1 = 0.55 - 0.1 = 0.45$$

$$y_2 = 0.55 + (0.1 \times 0.45) = 0.55 + 0.045 = 0.595$$

When $n=2$ in (4.1)

$$y_3 = y_2 + h f_2$$

$$f_2 = y_2 - x_2$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$$f_2 = 0.595 - 0.2 = 0.395$$

$$y_3 = 0.595 + (0.1 \times 0.395) = 0.595 + 0.0395 = 0.6345$$

When n=3

$$y_4 = y_3 + h f_3$$

$$f_3 = y_3 - x_3$$

$$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$$

$$f_3 = 0.6345 - 0.3 = 0.3345$$

$$y_4 = 0.6345 + (0.1 \times 0.3345) = 0.6345 + 0.03345 = 0.66795$$

When n=4

$$y_5 = y_4 + h f_4$$

$$f_4 = y_4 - x_4$$

$$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$$

$$f_4 = 0.66795 - 0.4 = 0.26795$$

$$y_5 = 0.66795 + (0.1 \times 0.26795) = 0.66795 + 0.026795 = 0.694745$$

When n=5

$$y_6 = y_5 + h f_5$$

$$f_5 = y_5 - x_5$$

$$x_5 = x_4 + h = 0.4 + 0.1 = 0.5$$

$$f_5 = 0.694745 - 0.5 = 0.194745$$

$$y_6 = 0.694745 + (0.1 \times 0.194745) = 0.694745 + 0.0194745 = 0.7142195$$

When n=6

$$y_7 = y_6 + h f_6$$

$$f_6 = y_6 - x_6$$

$$x_6 = x_5 + h = 0.5 + 0.1 = 0.6$$

$$f_6 = 0.7142195 - 0.6 = 0.1142195$$

$$y_7 = 0.7142195 + (0.1 \times 0.1142195) = 0.7142195 + 0.01142195 = 0.72564145$$

When n=7

$$y_8 = y_7 + h f_7$$

$$f_7 = y_7 - x_7$$

$$x_7 = x_6 + h = 0.6 + 0.1 = 0.7$$

$$f_7 = 0.72564145 - 0.7 = 0.02564145$$

$$y_8 = 0.72564145 + (0.1 \times 0.02564145) = 0.72564145 + 0.002564145 = 0.728205595$$

When n=8

$$y_9 = y_8 + h f_8$$

$$f_8 = y_8 - x_8$$

$$x_8 = x_7 + h = 0.7 + 0.1 = 0.8$$

$$f_8 = 0.728205595 - 0.8 = -0.071794405$$

$$y_9 = 0.728205595 + (0.1 \times -0.071794405) = 0.728205595 - 0.0071794405 = 0.721061545$$

When n=9

$$y_{10} = y_9 + h f_9$$

$$f_9 = y_9 - x_9$$

$$x_9 = x_8 + h = 0.8 + 0.1 = 0.9$$

$$f_9 = 0.721061545 - 0.9 = -0.178938455$$

$$y_{10} = 0.721061545 + (0.1 \times -0.178938455) = 0.721061545 - 0.0178938455 = 0.7031676995$$

When n=10

$$y_{11} = y_{10} + h f_{10}$$

$$f_{10} = y_{10} - x_{10}$$

$$x_{10} = x_9 + h = 0.9 + 0.1 = 1.0$$

$$f_{10} = 0.7031676995 - 1.0 = -0.2968323005$$

$$y_{11} = 0.7031676995 + (0.1 \times -0.2968323005) = 0.7031676995 - 0.02968323005 = 0.67348446945$$

Table 3 The comparison between the exact solution and that obtained by the explicit Adams method for step size, h=0.1 when k=1

<i>n</i>	<i>x_n</i>	<i>f_n</i>	<i>y_n</i>	<i>exact solution</i>	<i>absolute error</i>
0	0	0.5000	0.5500000	0.5	0.05
1	0.1	0.45000	0.5950000	0.547414	0.047586
2	0.2	0.395000	0.6345000	0.589299	0.045201
3	0.3	0.334500	0.6679500	0.625070	0.04288
4	0.4	0.267950	0.6947450	0.654088	0.040657

5	0.5	0.194745	0.7142195	0.675639	0.038580
6	0.6	0.1142195	0.72564145	0.688940	0.036701
7	0.7	0.02564145	0.728205595	0.693124	0.035132
8	0.8	-0.071794405	0.721061545	0.687229	0.033832
9	0.9	-0.178938455	0.7031676995	0.670198	0.032969
10	1.0	-0.2968323005	0.67348446945	0.640858	0.032625

To solve for k=2

$$y_{n+2} = y_{n+1} + h(f_n + f_{n+1}) \dots\dots\dots(4.2)$$

When n=0

$$y_2 = y_1 + h(f_0 + f_1)$$

where

$$y_1 = 0.55, h = 0.1, f_0 = 0.5, f_1 = 0.45$$

$$y_2 = 0.55 + 0.1(0.5 + 0.45) = 0.55 + 0.1(0.95) = 0.55 + 0.095 = 0.645$$

When n=1

$$y_3 = y_2 + h(f_1 + f_2)$$

where

$$y_2 = 0.645, f_1 = 0.45, f_2 = 0.395$$

$$y_3 = 0.645 + 0.1(0.45 + 0.395) = 0.645 + 0.0845 = 0.7295$$

When n=2

$$y_4 = y_3 + h(f_2 + f_3)$$

3.5. Advantages of the Adams Bashforth method over the 4th order Runge Kutta method

- The 4th order Runge Kutta method (4th RKM) requires 4 evaluations of *f* at each step, while the 4th order Adams Bashforth method (when past the beginning/starting values) requires only one and the predictor-corrector method just two. Consequently, for a given step size *h*, the last two strategies likely could be thought of as quicker than Runge Kutta.
- The basic polynomial estimate in various step method make it simple to rough (approximate) solution at point between the lattice/mesh point, should be this desirable (alluring).
- Linear Multistep Method have become famous generally on the grounds that its somewhat simple to access the mistake or the step size to control it.

3.6. Disadvantages of Adams Bashforth

A significant disadvantage of the multistep equation is that they are not self-beginning. In any case, the first order Adams Bashforth formula is self-beginning since it requires just data accessible from problem itself. Accordingly, in the Adams Bashforth, we should have progressive value of *f(x, y(x))* at equally separated point prior to using the formula.

A subsequent disadvantage is that the local discretization error is $O(h^5)$, the coefficient in the error term is bigger than the equations of the Runge Kutta type of a similar order.

4. Conclusion

In conclusion, I have considered the Ordinary Differential equations and the Linear Multistep Method, applied the Adams Bashforth scheme to a solving a simple ordinary differential equation. It is also concluded that while solving Linear Multistep Method the previous values are gotten before the present values can be solved.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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