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Remarks on fractional Hamiltonian formulation of discrete systems with examples

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Abstract

The Fractional Hamiltonian is used to investigate discrete systems in terms of Caputo's fractional derivatives. Three models have been introduced and studied in order to apply the formalism presented here. The obtained Hamilton's equations of motion are exactly in agreement with the classical Hamiltonian formulation equations.

Keywords: Caputo's Fractional Derivatives; Lagrangian; Hamiltonian Formalution; Euler-Lagrange Equations; Discrete Systems

1. Introduction

Fractional calculus is a generalization of usual calculus. In this branch of mathematics, meanings are established for integrals and derivatives of any non-integer (even complex) order, such as $\frac{d^{1/2} f(t)}{dt^{1/2}}$. It has started in 1695 when Leibniz presented his investigation of the derivative of order $\frac{1}{2}$. During the last decades it was used to study numerous fields of engineering and science [1-7]. Numerical investigation of fractional differential equations appeared in many researches [8-14], and it played an essential role in solving these equations numerically for several systems.

Fractional calculus has been used to reformulate the Euler– Lagrange problems fractionally. Riewe investigated non-conservative Lagrangian and Hamiltonian mechanics and for those cases formulated a version of the Euler–Lagrange equations (ELE's) [11]. Other researches work on fractional Lagrangian and Hamiltonian approaches, see [16–21] and the references therein.

In this paper we developed the fractional Hamiltonian equations of motion (FHEOM) for discrete systems in terms of Caputo's fractional derivatives. The paper is prepared as follows:

In Sect. 2, we discussed briefly the Caputo's fractional Lagrangian mechanics. In Sect. 3, we present the Caputo's fractional Hamiltonian formalism for discrete systems. In Sect. 4, illustrative examples are discussed using the formalism presented. Finally, we close this paper with a conclusion in Sec. 5.

2. Basic Tools for Caputo's Fractional Lagrangian

The left Caputo's fractional derivative (i.e., LCFD), and the right Caputo's fractional derivative (i.e., RCFD) read respectively as [1, 2]:

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$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} (\frac{d}{d\tau})^n f(\tau) d\tau . \tag{1} $$

$$ {}_t^cD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} (-\frac{d}{d\tau})^n f(\tau) d\tau . \tag{2} $$

Note that α is the order of the derivative such that $n-1 \le \alpha \prec n$, and it is not equal to zero.

When α goes to an integer then, these two equations turned to the following two classical derivatives:

$$ {}_a^cD_t^\alpha f(t) = (\frac{d}{dt})^\alpha f(t). \tag{3} $$

$$ {}_t^cD_b^\alpha f(t) = (-\frac{d}{dt})^\alpha f(t). \tag{4} $$

Now, let us consider the action of integral

$$ S = \int L(\eta, {}_a^cD_t^\alpha \eta, {}_t^cD_b^\beta \eta, t) dt. \tag{5} $$

The corresponding (ELE's) are then obtained as:

$$ \frac{\partial L}{\partial \eta} + {}_t^cD_b^\alpha \frac{\partial L}{\partial {}_a^cD_t^\alpha \eta} + {}_a^cD_t^\beta \frac{\partial L}{\partial {}_t^cD_b^\beta \eta} = 0. \tag{6} $$

As $\alpha, \beta \to 1$, we have ${}_a^cD_t^\alpha = \frac{d}{dt}$, and ${}_t^cD_b^\beta = -\frac{d}{dt}$, then (6) reduces to the classical ELE's.

3. Caputo's Fractional Hamiltonian of Discrete Systems

Consider the Lagrangian of discrete systems that depends on the Caputo's fractional derivatives of coordinates with the following form:

$$ L = L(\eta, {}_a^cD_t^\alpha \eta, {}_t^cD_b^\beta \eta, t). \tag{7} $$

with $0 \prec \alpha, \beta \prec 1$.

Introducing the generalized momenta:

$$ P_\alpha = \frac{\partial L}{\partial {}_a^cD_t^\alpha \eta}; $$

$$ P_\beta = \frac{\partial L}{\partial {}_t^cD_b^\beta \eta}. \tag{8} $$

Thus, the Hamiltonian depending on Caputo's fractional derivatives can be written as:

$$ H = P_\alpha ({}_a^cD_t^\alpha \eta) + P_\beta ({}_t^cD_b^\beta \eta) - L. \tag{9} $$

Taking the total differential of the above Hamiltonian:

$$dH = P_\alpha (d\,{}_a^cD_t^\alpha \eta) + dP_\alpha ({}_a^cD_t^\alpha \eta) + P_\beta (d\,{}_t^cD_b^\beta \eta) + dP_\beta ({}_t^cD_b^\beta \eta) - \frac{\partial L}{\partial \eta} d\eta - \frac{\partial L}{\partial\, {}_a^cD_t^\alpha \eta} d\,{}_a^cD_t^\alpha \eta$$

$$- \frac{\partial L}{\partial\, {}_t^cD_b^\beta \eta} d\,{}_t^cD_b^\beta \eta - \frac{\partial L}{\partial t} dt. \qquad (10)$$

Substituting (8) into (10), we get:

$$dH = dP_\alpha ({}_a^cD_t^\alpha \eta) + dP_\beta ({}_t^cD_b^\beta \eta) - \frac{\partial L}{\partial \eta} d\eta - \frac{\partial L}{\partial t} dt. \qquad (11)$$

Making use of (6), then (11) becomes:

$$dH = dP_\alpha ({}_a^cD_t^\alpha \eta) + dP_\beta ({}_t^cD_b^\beta \eta) - [{}_t^cD_b^\alpha P_\alpha + {}_a^cD_t^\beta P_\beta] d\eta - \frac{\partial L}{\partial t} dt. \qquad (12)$$

This means that the Hamiltonian is a function of

$$H = (\eta, P_\alpha, P_\beta, t). \qquad (13)$$

The total differential of (13) is:

$$dH = \frac{\partial H}{\partial \eta} d\eta + \frac{\partial H}{\partial P_\alpha} dP_\alpha + \frac{\partial H}{\partial P_\beta} dP_\beta - \frac{\partial H}{\partial t} dt. \qquad (14)$$

Comparing (14) with (11), one gets the following FHEOM:

$$\frac{\partial H}{\partial P_\alpha} = {}_a^cD_t^\alpha \eta\,;$$

$$\frac{\partial H}{\partial P_\beta} = {}_t^cD_b^\beta \eta\,;$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\,;$$

and finally,

$$\frac{\partial H}{\partial q_\alpha} = ({}_t^cD_b^\alpha) P_\alpha + ({}_a^cD_t^\beta) P_\beta. \qquad (15)$$

4. Illustrative Examples

In this section, three examples (Mathematical and Physical) will be investigated using the formalism presented in Sec. 3 above.

4.1. Firstly, let us consider the following Mathematical example as a first discrete system

$$S = \int_0^1 L dt \,.$$

where $L = \frac{1}{2}({}_0^cD_t^\alpha \eta)^2 + \frac{1}{2}({}_t^cD_1^\beta \eta)^2 + ({}_0^cD_t^\alpha \eta)({}_t^cD_1^\beta \eta)$.

The generalized momenta (8) read:

$$P_\alpha = {}_0^cD_t^\alpha \eta + {}_t^cD_1^\beta \eta \,;$$

$$P_\beta = {}_t^cD_1^\beta \eta + {}_0^cD_t^\alpha \eta \,.$$

Then according (9) the Hamiltonian can be calculated as:

$$H = \frac{1}{2} P_\alpha^2 = \frac{1}{2} P_\beta^2 \,.$$

As a result the HEM (15) read:

$$P_\alpha = {}_0^cD_t^\alpha \eta \Rightarrow [{}_0^cD_t^\alpha \eta + {}_t^cD_1^\beta \eta] = {}_0^cD_t^\alpha \eta \,;$$

$$P_\beta = {}_t^cD_1^\beta \eta \Rightarrow [{}_0^cD_t^\alpha \eta + {}_t^cD_1^\beta \eta] = {}_t^cD_1^\beta \eta \,;$$

$$0 = [{}_t^cD_1^\beta \eta + {}_0^cD_t^\alpha \eta][{}_0^cD_t^\alpha \eta + {}_t^cD_1^\beta \eta] \,.$$

This result is in exact agreement with that obtained by Agrawal [17].

4.2. Secondly, we study the following physical model

$$S = \int_0^1 L dt \,.$$

where $L = \frac{1}{2}({}_0^cD_t^\alpha \eta)^2$.

This fractional model is the analog of free motion in one- dimensional space.

The generalized momenta (8) read:

$$P_\alpha = {}_0^cD_t^\alpha \eta \,;$$

$$P_\beta = 0 \,.$$

Now, making use of (9) the Hamiltonian is calculated as:

$$H = \frac{1}{2} P_\alpha^2 .$$

The HEM (15) read:

$$P_\alpha = {}_0^c D_t^\alpha \eta ;$$

$${}_t^c D_1^\beta \eta = 0 ;$$

$$0 = ({}_t^c D_1^\alpha \eta)({}_0^c D_t^\alpha \eta) .$$

As $\alpha \to 1$ the last equation reads the classical equation of motion for a free motion in a one dimensional space, i.e., $\frac{d^2\eta}{dt^2} = 0$.

4.3. Finally, we study the following physical model

$$S = \int_0^1 L dt .$$

where $L = \frac{1}{2} m ({}_0^c D_t^\alpha \eta)^2 + \frac{1}{2} k \eta^2$.

This is the fractional analog of the classical Harmonic Oscillator.

The generalized momenta (8) read:

$$P_\alpha = m\, {}_0^c D_t^\alpha \eta ;$$

$${}_t^c D_1^\beta \eta = 0 .$$

As a result the Hamiltonian is:

$$H = \frac{P_\alpha^2}{2m} - \frac{1}{2} k \eta^2 .$$

The HEM read:

$$P_\alpha = m\, {}_0^c D_t^\alpha \eta ;$$

$$0 = {}_t^c D_1^\beta \eta ;$$

$$-k\eta = m({}_t^c D_1^\alpha \eta)({}_0^c D_t^\alpha \eta) .$$

Again, as $\alpha \to 1$ the last equation reads the classical equation of motion for a Harmonic Oscillator, i.e.

$$m \frac{d^2\eta}{dt^2} - k\eta = 0$$

5. Conclusion

Using the Caputo's fractional derivatives, the fractional Hamiltonian of discrete systems has been constructed. The Hamilton's equations have been attained for discrete systems, where some examples are studied. We note that as $\alpha \to 1$ then the fractional Hamilton's equations reduced to the classical equations.

Finally, it is noted that the Hamiltonian formulation is in exact agreement with the Lagrangain formulation

Compliance with ethical standards

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Disclosure of conflict of interest

The authors declare no conflict of interest.

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Author's Short Biography

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