(RESEARCH ARTICLE)

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Active learning strategy: Computer aided numerical class project on pole-zero plot and transfer function of five low pass filter approximation functions

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Abstract

The conversion of the concept and the teaching of approximation function for analog low pass filter to computer aided numerical class project as an active learning strategy is presented. Governing equations for five approximation functions are presented and discussed. It is recommended that students are tasked to develop computer programs in any convenient computer programming language to make the pole-zero plot, obtain the transfer function, and plot the frequency response of low pass filters. Steps for such programs are suggested. Typical programs were developed and tested. The pole-zero plots and frequency response graphs generated are good illustrative and innovative teaching aids. Linear-phase response over the passband is a characteristic of Bessel approximation function but suffers less amplitude discrimination in the stopband. Chebyshev and Inverse Chebyshev approximation functions have the same order requirement which is usually greater than that of Elliptic approximation function but less than that of Butterworth approximation function. Elliptic approximation function is the best choice although it is more complex than the others. Through this computer aided numerical class project, the authors themselves gained more insight in the subject matter; certainly, students will learn actively.

Keywords: Active learning; Filter design; Approximation functions; Pole-zero plot; Transfer function; Frequency response; Teaching aids

1. Introduction

Attention is shifting from passive learning or teacher centered learning to active learning or student centered learning. Active learning requires the students to be actively engaged in the learning process. Active learning attracts commitment, focus, and attention of students. No dull moment in active learning classes. Active learning strategies are classroom-based activities designed to engage students in their learning through answering questions, solving problems, discussing content, or teaching others, individually or in groups [1, 2, 3].

Active learning involves setting goals and objective learning outcomes. There is reciprocity and cooperation among peers. There is feedback via meaningful interactions between learners, peers and educators. Students learn faster and are able to retain and apply the knowledge acquired. Active learning has been found to be effective in all disciplines [4, 5, 6, 7, 8, 9]. Passive learning enables students to learn at the lower levels of Bloom's taxonomy of Fig. 1 [10]. Active learning achieves students' learning at both the lower and higher levels of Bloom's taxonomy.

Engineering Education transform engineering students from being novices to being experts [11]. With respect to the concept students learn, students are groomed to remember, to understand, to be able to apply, to be able to analyse, to

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be able to evaluate and to be able to create. Part of the strategy to achieve these objectives is to create class projects or opportunity for the students to solve problems. Development of innovative teaching aids has also been recommended [12].
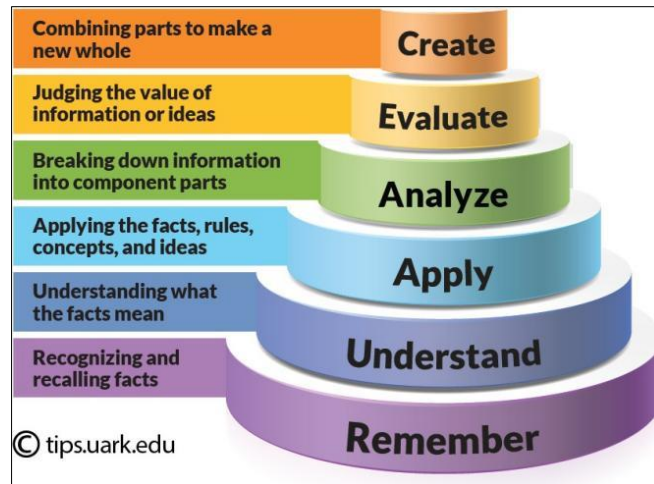
Figure 1 Bloom's Taxonomy [10]

In this paper, the concept of approximation function for analog low pass filter is converted to a computer aided numerical class project as an active learning strategy. Diagrams are also generated as teaching aids for the concept. The class project is based on computer programming and Engineering Mathematics. Why Engineering Mathematics? Why Computer Programming? Asking questions is part of active learning strategies. Making students to find out "what", "why", and "how" ensures students' active participation and active learning in class. What is the concept and what is the concept all about? Why must the student learn the concept? The "why" is about the significance, the relevance, and applications of the concept. "How" is the technical know-how of the concept.

Engineering components, devices, and system have mathematical expressions relating their parameters, inputs and outputs. These components, devices, and systems have ordinary or differential or difference equations that describe them. The branch of mathematics consisting of mathematical methods and techniques which are applied in engineering and industry is known as engineering mathematics.

The analysis and design of engineering systems requires computations. At times, these computations may be complex, tedious, cumbersome, repetitive or iterative. The assistance of a digital computer is helpful in such situations. Computer programming is the design of a sequence of instructions in a particular computer programming language to be executed by a digital computer to solve a given problem.

What is a filter? Why is filtering necessary? A filter is a frequency selective network which passes some frequencies and blocks some frequencies. Filtering is necessary to extract a desired signal from the available mixture of signals. Analog filter processes analog signals while digital filter processes discrete time signals. A low pass filter passes low frequency signals and blocks high frequency signals.

What are approximation functions and why are they necessary? The ideal low pass filter frequency response of Fig. 2 is not realizable with components or combination of components such as resistors, inductors, capacitors, and operational amplifiers (Op Amp). An approximation function provides a realizable frequency response of Fig. 3 which is approximate to the ideal frequency response. The filter function or response H(s) is the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal with all initial conditions assumed to be zero.

Fig. 4 shows the first order and second order passive networks which have approximate low pass filter responses. Suppose that R_1 and R_2 are 1 MΩ resistors. Suppose C_1 and C_2 are 1 µF capacitors. The Analysis of the first order circuit of Fig. 4(a) leads to Eqns. (1) and (2) in the time domain and the frequency domain respectively [13,14,15]. Eqn. (2) leads to the transfer function H(s), H(jw), and the magnitude response |H(w)| of Eqns. (3), (4), and (5) respectively. The frequency response for the first order network of Fig. 4 (a) is shown in Fig. 5 and is an approximation to that of the ideal low pass filter as in Fig. 3. This network has a pole at $s = -1$ and a zero at infinity.
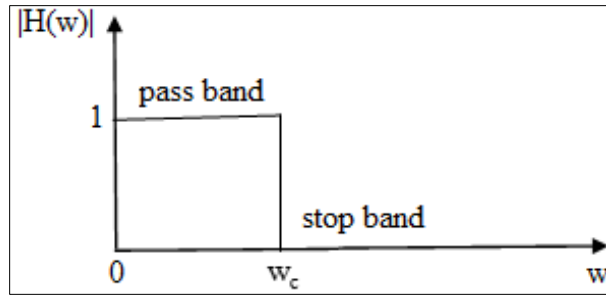
Figure 2 Ideal low pass filter frequency response
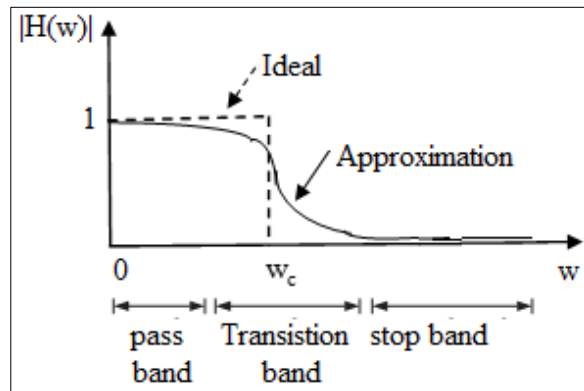
Figure 3 Approximate low pass filter frequency response

(a) First order network

(b) Second order network
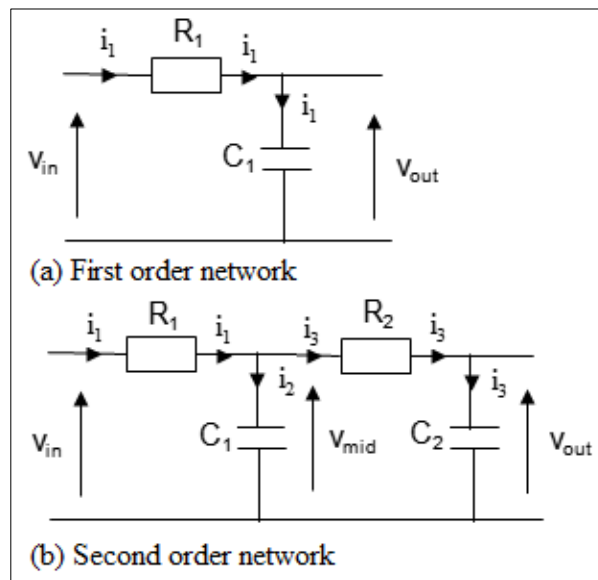
Figure 4 Passive network implementation of low pass filters

$$v_{in}(t) = R_1 C_1 \frac{dv_{out}}{dt} + v_{out}(t) \qquad \text{.....................} \qquad (1)$$

$$V_{in}(s) = R_1 C_1 s V_{out}(s) + V_{out}(s) \qquad \text{.....................} \qquad (2)$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{R_1 C_1 s + 1} = \frac{1}{s+1} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$H(w) = \frac{1}{jw+1} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$|H(w)| = \frac{1}{\sqrt{(1+w^2)}} \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$
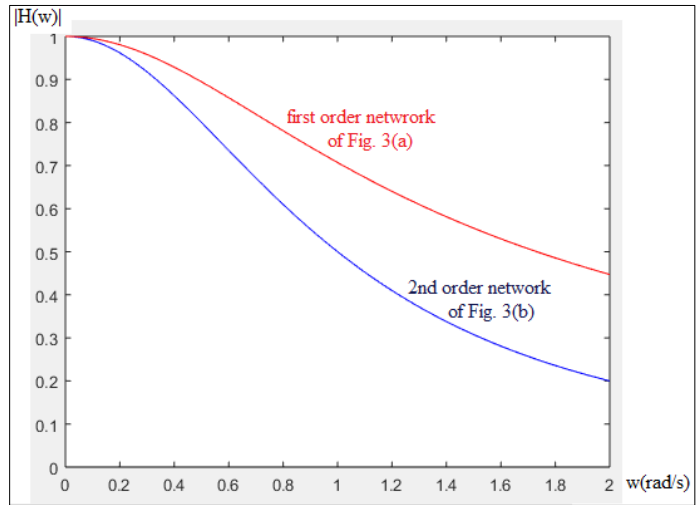
Figure 5 The frequency response of the first and second order networks of Fig. 4

What is a pole? What is a zero? A pole of a transfer function is the value of s which makes the transfer function tends to infinity. A zero of a transfer function is the value of s which makes the transfer function tends to zero. The pole-zero plot of a network defines the properties and characteristics of that network. A pole is represented by x while a zero is represented by o on the s-plane. A zero at infinity is not indicated on the pole-zero plot.

The Analysis of the second order circuit of Fig. 4(b) leads to Eqns. (6) and (7) in the time domain and the frequency domain respectively [13, 14, 15]. Eqn. (7) leads to the transfer function H(s), H (jw), and the magnitude response |H(w)| of Eqns. (8), (9), and (10) respectively. The frequency response for the second order network of Fig. 4(b) is also shown in Fig. 5 and is an approximation to that of the ideal low pass filter as in Fig. 3. This network has two poles at $s = -1$ and a zero at infinity. The higher the order of the network, the higher the attenuation in the stopband and the narrower the transition band.

$$v_{in}(t) = R_1 C_1 R_2 C_2 \frac{d^2 v_{out}}{dt^2} + (R_1 C_1 + R_2 C_2)\frac{dv_{out}}{dt} + v_{out}(t) \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$V_{in}(s) = s^2 V_{out}(s) + 2s V_{out}(s) + V_{out}(s) \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{s^2 + +2s + 1} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$H(w) = \frac{1}{j2w + (1-w^2)} \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

$$|H(w)| = \frac{1}{\sqrt{[(1-w^2)^2 + 4w^2]}} \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

The approximation functions are therefore functions which are approximate to the frequency response of an ideal filter and which can be realized with passive and or active circuit components. Approximation functions introduce transition band in between the passband and the stopband as illustrated in Fig. 3. Five approximation functions are considered in this work.

Fig. 6 gives the performance specification of a non-ideal low pass filter. An explicit specification of the performance requirement is provided in terms of the frequency response characteristics, the desired amplitude and/or phase responses, and their tolerances.
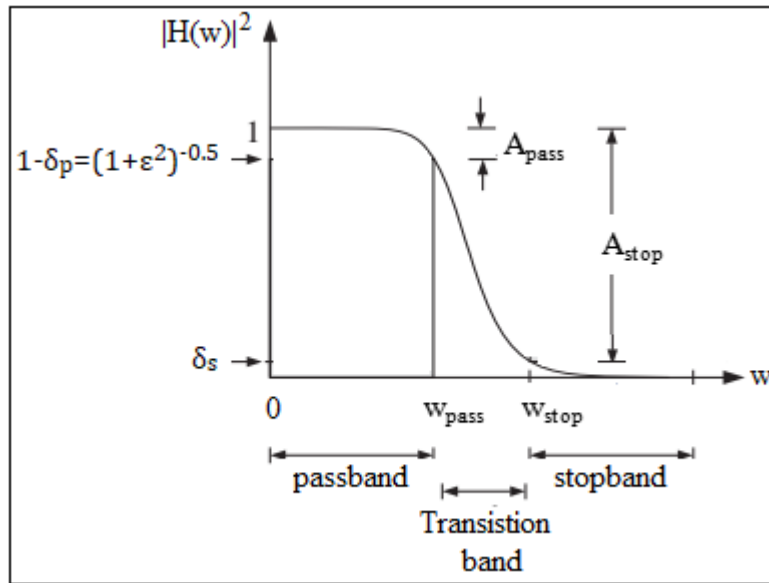
Figure 6 Low pass filter specifications

ε is the passband ripple parameter, δ_p is the passband deviation, δ_s is stopband deviation, w_{pass} is the passband edge frequency, and w_{stop} is the stopband edge frequency. The band edge frequencies are in standard frequency unit of rad/sec and can also be expressed in Hertz (Hz). Passband deviation and stopband deviation may be expressed as ordinary numbers or in decibels as described in Eqns. (11) and (12) respectively [16]. A$_{pass}$ and A$_{stop}$ are the passband Gain and the stopband Gain in decibels respectively.

$$A_{pass} = 10log_{10}(1 + \varepsilon^2) = -20log_{10}(1 - \delta_p) \qquad .... \qquad (11)$$

$$A_{stop} = -20log_{10}(\delta_s) \qquad .... \qquad (12)$$

2. Methodology

2.1. Butterworth Approximation

The Butterworth approximation function is often called the maximally flat response because no other approximation has a smoother transition through the passband to the stopband [17]. The phase response also is very smooth, which is important when considering distortion. The lowpass Butterworth polynomial has an all-pole transfer function with no finite zeros present. It is the approximation method of choice when low phase distortion and moderate selectivity are required.

Eqn. (13) gives the Butterworth approximation's magnitude response where ω_{pass} is the passband edge frequency for the filter, n is the order of the approximation function, and ε is the passband Gain adjustment factor [18]. The transfer functions carry subscripts for identification. In this case, the subscript B indicates a Butterworth filter, and n indicates an nth-order transfer function.

$$|H_{B,n}(jw)| = \frac{1}{\sqrt{1+\varepsilon^2.(w/w_{pass})^{2n}}} \qquad .... \qquad (13)$$

where

$$\varepsilon = \sqrt{10^{-0.1A_{pass}} - 1} \qquad .... \qquad (14)$$

With ε set to 1 and ω_{pass} set to 1, the filter will have a Gain of $1/\sqrt{2}$ or −3.01 dB at the normalized passband edge frequency of 1 rad/sec. The Gain is simply 20Log[|H(jw)|].The Butterworth approximation has a number of interesting properties. First, the response will always have unity Gain (0 dB) at $\omega = 0$, no matter what value is given to ε. However, the Gain at the normalized passband edge frequency of $\omega = 1$ rad/sec will depend on the value of ε. Furthermore, the response Gain decreases by a factor of $-20n$ dB per decade of frequency change [19]. The order of the Butterworth filter to meet given specifications is given by Eqn. (15).

$$n_B = \frac{\log[(10^{-0.1A_{stop}}-1)/(10^{-0.1A_{pass}}-1)]}{2\log(w_{stop}/w_{pass})} \qquad \text{......................} \qquad (15)$$

The transfer function can be developed from the filter's poles and zeros in the left half of the s-plane. Butterworth filter is an all-pole filer with no zero. The poles for a Butterworth approximation function are equally spaced around a semicircle in the s-plane and are symmetrical about the real axis. The number of poles is equal to the order number and the radius of the semicircle is given by Eqn. (16). Once the radius of the circle is known, the pole positions are determined by calculating the necessary angles $\theta_0, \theta_1, \ldots, \theta_m$ in the second quadrant in accordance with Eqns. (17) and (18) for even-order and odd-order respectively. Any complex pole in the second quadrant has a conjugate pair in the third quadrant. The real part and the imaginary part of the pole for each θ_m are determined by Eqns. (19) and (20) respectively. For odd-order functions, there is a pole on the real axis at $\sigma = -R$.

$$R = \varepsilon^{-1/n} \qquad \text{......................} \qquad (16)$$

$$\theta_m = \frac{\pi(2m+n+1)}{2n}, \text{ m = 0, 1,..., (n/2)-1 (n even)} \qquad \text{......................} \qquad (17)$$

$$\theta_m = \frac{\pi(2m+n+1)}{2n}, \text{ m = 0, 1,..., [(n-1)/2]-1 (n odd)} \qquad \text{......................} \qquad (18)$$

$$\sigma_m = R\cos(\theta_m) \quad \text{......................} \qquad (19)$$

$$w_m = R\sin(\theta_m) \quad \text{......................} \qquad (20)$$

The complete Butterworth approximation transfer function can be determined from a combination of a first-order factor (for odd orders) presented in Eqn. (21) and quadratic factors presented in Eqn. (22) and explained in Eqns. (23) and (24). Each of these factors will have a constant in the numerator to adjust the Gain to unity (0 dB) at $\omega = 0$. The complete Butterworth transfer function can now be defined as in Eqns. (25) and (26) for even-order and odd-order respectively.

$$H_0(S) = \frac{R}{S+R} \qquad \text{......................} \qquad (21)$$

$$H_m(S) = \frac{B_{2m}}{S^2 + B_{1m}S + B_{2m}} \qquad \text{......................} \qquad (22)$$

where

$$B_{1m} = -2\sigma_m \qquad \text{......................} \qquad (23)$$

$$B_{2m} = \sigma_m^2 + w_m^2 \text{......................} \qquad (24)$$

$$H_{B,n}(S) = \frac{\prod_{m=0}^{(n/2)-1}(B_{2m})}{\prod_{m=0}^{(n/2)-1}(S^2 + B_{1m}S + B_{2m})}, \quad \text{(n even)......................} \qquad (25)$$

$$H_{B,n}(S) = \frac{R\prod_{m=0}^{[(n-1)/2]-1}(B_{2m})}{(S+R)\prod_{m=0}^{[(n-1)/2]-1}(S^2 + B_{1m}S + B_{2m})}, \text{(n odd)} \qquad \text{......................} \qquad (26)$$

As a class project, students should be tasked to develop a computer program to make the pole-zero plot, obtain the transfer function, and plot the frequency response for Butterworth low pass filter in any computer programming language of their choice. In this work, a program was developed in Matlab working environment [20,21]; the steps are presented in Table 1.

2.2. Chebyshev Approximation

The Chebyshev polynomial is a powerful function in approximation theory. Although the function is a polynomial, it is best defined and developed in terms of trigonometric functions [22]. The Chebyshev approximation function also has an all-pole transfer function. The Chebyshev filter allows variation or ripple in the passband of the filter. The transition characteristics of the Chebyshev is steeper than that of the Butterworth. However, the phase response is not as linear as that of Butterworth, and therefore if low phase distortion is a priority, the Chebyshev approximation may not be the best choice [17].

Table 1 Program to make the pole-zero diagram, obtain the transfer function, and plot the frequency response of Butterworth low pass filter

Step	Description
1	Input A_{pass}, A_{stop}, ω_{pass}, and ω_{stop}
2	Calculate ε, n, and R using Eqns. (14), (15), and (16) respectively; n is rounded up to the next higher whole number using the code "ceil" in Matlab [20].

3	Plot a circle of radius R on the s-plane. The following sub-program is suggested.
	nn1=0;
	for t=0:2*pi/10000:2*pi
	 nn1=nn1+1;
	 y(nn1)=R*sin(t);
	 x(nn1)=R*cos(t);
	end
	plot(x,y,'m--')

4	If n is odd, get the real pole at $\sigma = -R$. Plot the pole on the s-plane.
5	Set m=-1.
6	Increase m by 1.
7	Calculate θ_m, σ_m, w_m, B_{1m}, and B_{2m}, using Eqns. (17 or 18), (19), (20), (23) and (24) respectively. Plot the pole and its conjugate pair on the s-plane.
8	Is $m < (n/2) - 1$ for even order or is $m < [(n-1)/2] - 1$ for odd order? If Yes, go back to Step 6. If No, proceed to Step 9.
9	Obtain the transfer function using Eqn. (25) or (26).
10	Plot the frequency response

The magnitude response function for the Chebyshev approximation is given by Eqn. (27). $Cn(\omega)$, the Chebyshev polynomial of the first kind of degree n, is defined as in Eqn. (28). The order of the Chebyshev filter is given by Eqn. (29).

$$|H_{C,n}[j(w/w_0)]| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(w/w_0)}} \quad \text{.....................} \quad (27)$$

Where the definition of ε is again as given by Eqn. (14).

$$C_n(w) = \cosh[n \cosh^{-1}(w)], w > 0 \quad \text{....................} \quad (28)$$

$$n_C = \frac{\cosh^{-1}[\sqrt{(10^{-0.1 A_{stop}}-1)/(10^{-0.1 A_{pass}}-1)}]}{\cosh^{-1}(w_{stop}/w_{pass})} \quad \text{....................} \quad (29)$$

The poles for a Chebyshev approximation function are located on an ellipse. The ellipse is centered at the origin of the s-plane with its major axis along the jω axis with intercepts of ± cosh(D), while the minor axis is along the real axis with intercepts of ± sinh(D). The variable D is defined as

$$D = \frac{\sinh^{-1}\varepsilon^{-1}}{n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (30)$$

The pole positions are determined by calculating the necessary angles $\theta_0, \theta_1, \ldots, \theta_m$ in the first quadrant in accordance with Eqns. (31) and (32) for even-order and odd-order respectively. The real part and the imaginary part of the pole for each θ_m are determined by Eqns. (33) and (34) respectively. The negative sign in Eqn. (33) translates the angles in the first quadrant to the second quadrant. Any complex pole in the second quadrant has a conjugate pair in the third quadrant. For odd-order functions, there is a pole on the real axis at $\sigma = -\sinh(D)$.

$$\theta_m = \frac{\pi(2m+1)}{2n}, \text{m=0,1,…,(n/2)-1 (n even)} \quad \ldots\ldots\ldots\ldots\ldots \qquad (31)$$

$$\theta_m = \frac{\pi(2m+1)}{2n}, \text{m=0,1,…,[(n-1)/2]-1 (n odd)} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (32)$$

$$\sigma_m = -\sinh(D)\sin(\theta_m) \qquad \ldots\ldots\ldots\ldots\ldots \qquad (33)$$

$$w_m = \cosh(D)\cos(\theta_m) \quad \ldots\ldots\ldots\ldots\ldots \qquad (34)$$

The complete Chebyshev approximation transfer function can be determined from a combination of a first-order factor (for odd orders) presented in Eqn. (35) and quadratic factors presented in Eqn. (36) and explained in Eqns. (37) and (38). Each of these factors will have a constant in the numerator to adjust the Gain to unity (0 dB) at ω = 0. Furthermore, a Gain constant G given by Eqn. (39) must be included for even-order transfer functions to adjust the Gain to unity (0 dB) at ω = 0. The complete Chebyshev transfer function can now be defined as shown in Eqns. (40) and (41) for even-order and odd-order respectively.

$$H_0(S) = \frac{\sinh(D)}{S + \sinh(D)} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (35)$$

$$H_m(S) = \frac{B_{2m}}{S^2 + B_{1m}S + B_{2m}} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (36)$$

$$B_{1m} = -2\sigma_m \qquad \ldots\ldots\ldots\ldots\ldots \qquad (37)$$

$$B_{2m} = \sigma_m^2 + w_m^2 \quad \ldots\ldots\ldots\ldots\ldots \qquad (38)$$

$$G = 10^{0.05 A_{pass}} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (39)$$

$$H_{C,n}(S) = \frac{G\prod_{m=0}^{(n/2)-1}(B_{2m})}{\prod_{m=0}^{(n/2)-1}(S^2 + B_{1m}S + B_{2m})}, (n\ even) \ldots\ldots\ldots\ldots\ldots \qquad (40)$$

$$H_{C,n}(S) = \frac{\sinh(D)\prod_{m=0}^{[(n-1)/2]-1}(B_{2m})}{(S+\sinh(D))\prod_{m=0}^{[(n-1)/2]-1}(S^2 + B_{1m}S + B_{2m})}, (n\ odd) \qquad \ldots\ldots\ldots\ldots\ldots \qquad (41)$$

As a class project, students should be tasked to develop a computer program to make the pole-zero plot, obtain the transfer function, and plot the frequency response for Chebyshev low pass filter. In this work, a program was developed in Matlab working environment [20, 21]; the steps are presented in Table 2.

2.3. Inverse Chebyshev Approximation

The Inverse Chebyshev approximation function, also called the Chebyshev type II approximation function, is a rational approximation with both poles and zeros in its transfer function. This approximation has a smooth, maximally flat response in the passband but has ripple in the stopband caused by the zeros of the transfer function [23]. The Inverse Chebyshev approximation provides better transition characteristics than the Butterworth filter and better phase response than the standard Chebyshev. The Inverse Chebyshev approximation requires more computations to design.

The name "Inverse Chebyshev" is well-deserved as many of the computations are based on inverse or reciprocal values from the standard computations of Chebyshev. The magnitude frequency response function is shown in Eqn. (42). $Cn(\omega)$ still represents the Chebyshev polynomial of the first kind of degree n as defined in (28) but $\omega o/\omega$ is used in Eqn. (42) instead of $\omega/\omega o$ used in Eqn. (28). ε_i is used instead of ε. ε_i is defined in Eqn. (43). The calculation of the order for

an Inverse Chebyshev filter is given by Eqn. (44) but it is exactly the same as for the standard Chebyshev filter given by Eqn. (29).

Table 2 Program to make the pole-zero plot, obtain the transfer function, and plot the frequency response of Chebyshev low pass filter

Step	Description
1	Input A_{pass}, A_{stop}, ω_{pass}, and ω_{stop}
2	Calculate ε, n, and D using Eqns. (14), (29), and (30) respectively; n is rounded up to the next higher whole number using the code "ceil" in Matlab [20].

3	Plot an ellipse with major axis $\pm \cosh(D)$ along the jω axis and minor axis $\pm \sinh(D)$ along the real axis on the s-plane. The following sub-program is suggested.
	nn1=0;
	for t=0:2*pi/10000:2*pi
	 nn1=nn1+1;
	 x(nn1)=-sinh(D)*sin(t);
	 y(nn1)=cosh(D)*cos(t);
	end
	plot(x,y,'m--')

4	If n is odd, get the real pole at $\sigma = -sinh(D)$. Plot the pole on the s-plane. If n is even, obtain G from Eqn. (39).
5	Set m=-1.
6	Increase m by 1.
7	Calculate θ_m, σ_m, w_m, B_{1m}, and B_{2m}, using Eqns. (31 or 32), (33), (34), (37) and (38) respectively. Plot the pole and its conjugate pair on the s-plane.
8	Is $m < (n/2) - 1$ for even order or is $m < [(n-1)/2] - 1$ for odd order? If Yes, go back to Step 6. If No, proceed to Step 9.
9	Obtain the transfer function using Eqn. (40) or (41).
10	Plot the frequency response

$$|H_{I,n}[j(w/w_0)]| = \frac{\sqrt{\varepsilon_i^2 C_n^2(w_0/w)}}{\sqrt{1+\varepsilon_i^2 C_n^2(w_0/w)}} \qquad \dots\dots\dots\dots\dots \qquad (42)$$

where

$$\varepsilon_i = \frac{1}{\sqrt{10^{-0.1A_{stop}}-1}} \qquad \dots\dots\dots\dots\dots \qquad (43)$$

$$n_I = \frac{\cosh^{-1}[\sqrt{(10^{-0.1A_{stop}}-1)/(10^{-0.1A_{pass}}-1)}]}{\cosh^{-1}(w_{stop}/w_{pass})} \qquad \dots\dots\dots\dots\dots \qquad (44)$$

The determination of the pole locations for the normalized Inverse Chebyshev approximation is based on techniques similar to those used for the standard normalized Chebyshev approximation. The Inverse Chebyshev poles are the reciprocals of the standard Chebyshev poles.

For example, if there exists a standard Chebyshev pole at

$$p = \sigma + j\omega \qquad \text{.....................} \qquad (45)$$

Then, the reciprocal of p gives the Inverse Chebyshev pole position as

$$p^{-1} = \frac{\sigma - jw}{(\sigma + jw)(\sigma - jw)} = \frac{\sigma}{\sigma^2 + w^2} - j\frac{w}{\sigma^2 + w^2} \qquad \text{.....................} \qquad (46)$$

If a pole's distance from the origin is greater than one, the reciprocal's distance will be less than one and vice versa. Furthermore, the position of the pole is reflected across the real axis; if the original pole position is in the second quadrant, the reciprocal is located in the third quadrant. The mathematical equations necessary to determine the pole locations for the Inverse Chebyshev approximation function are similar to those for the standard Chebyshev case. First, D_i is defined as

$$D_i = \frac{\sinh^{-1}(\varepsilon_i^{-1})}{n} \qquad \text{.....................} \qquad (47)$$

The pole locations in the second quadrant are obtainable using Eqns. (48) to (51). The prime values must still be inverted to get the actual Inverse Chebyshev poles as in Eqns. (52) and (53). For odd-order functions, there is a pole on the real axis at $\sigma' = -\sinh(D_i)$.

$$\sigma'_m = -\sinh(D_i)\sin(\theta_m) \qquad \text{.....................} \qquad (48)$$

$$w'_m = \cosh(D_i)\cos(\theta_m) \qquad \text{.....................} \qquad (49)$$

$$\theta_m = \frac{\pi(2m+1)}{2n}, m = 0,1,\dots,(n/2) - 1 \, (\text{n even}) \qquad \text{.....................} \qquad (50)$$

$$\theta_m = \frac{\pi(2m+1)}{2n}, m = 0,1,\dots,[(n-1)/2] - 1 \, (\text{n odd}) \qquad \text{.....................} \qquad (51)$$

$$\sigma_m = \frac{\sigma'_m}{\sigma'^2_m + w'^2_m} \qquad \text{.....................} \qquad (52)$$

$$w_m = \frac{-w'_m}{\sigma'^2_m + w'^2_m} \qquad \text{.....................} \qquad (53)$$

A pair of complex conjugate poles will create a pair of finite complex conjugate zeros on the $j\omega$ axis. These zeros are given by Eqns. (54) and (55). A real pole will create a zero at infinity.

$$\sigma_{zm} = 0.0 \qquad \text{.....................} \qquad (54)$$
$$\omega_{zm} = \sec(\theta_m) \qquad \text{.....................} \qquad (55)$$

The complete Inverse Chebyshev approximation transfer function can be determined from a combination of a first-order factor (for odd orders) presented in Eqn. (56) and quadratic factors presented in Eqn. (57) and explained in Eqns. (58) to (61). Each of these pole factors will have a constant in the numerator and each of these zero factors will have a constant in the denominator to adjust the Gain to unity (0 dB) at $\omega = 0$. The complete Inverse Chebyshev transfer function can now be defined as shown in Eqns. (62) and (63) for even-order and odd order-respectively.

$$H_0(S) = \frac{[\sinh D_i]^{-1}}{S + [\sinh D_i]^{-1}} \qquad \text{.....................} \qquad (56)$$

$$H_m(s) = \frac{B_{2m}(S^2 + A_{1m}S + A_{2m})}{A_{2m}(S^2 + B_{1m}S + B_{2m})} \qquad \text{.....................} \qquad (57)$$

$$B_{1m} = -2\sigma_m \qquad \text{.....................} \qquad (58)$$

$$B_{2m} = \sigma_m^2 + w_m^2 \qquad \text{.....................} \qquad (59)$$

$$A_{1m} = -2\sigma_{zm} = 0.0 \qquad \text{.....................} \qquad (60)$$

$$A_{2m} = \sigma_{zm}^2 + w_{zm}^2 = w_{zm}^2 \qquad \text{.....................} \qquad (61)$$

$$H_{I,n}(S) = \frac{\prod_{m=0}^{(n/2)-1}(B_{2m})\prod_{m=0}^{(n/2)-1}(S^2+A_{1m}S+A_{2m})}{\prod_{m=0}^{(n/2)-1}(A_{2m})\prod_{m=0}^{(n/2)-1}(S^2+B_{1m}S+B_{2m})}, \text{ (n even)} \quad \text{....................} \quad (62)$$

$$H_{I,n}(S) = \frac{[\sinh D_i]^{-1}\prod_{m=0}^{[(n-1)/2]-1}(B_{2m})\prod_{m=0}^{[(n-1)/2]-1}(S^2+A_{1m}S+A_{2m})}{(S+[\sinh D_i]^{-1})\prod_{m=0}^{[(n-1)/2]-1}(A_{2m})\prod_{m=0}^{[(n-1)/2]-1}(S^2+B_{1m}S+B_{2m})}, \text{ (n odd)} \quad \text{....................} \quad (63)$$

As a class project, students should be tasked to develop a computer program to make the pole-zero plot, obtain the transfer function, and plot the frequency response for Inverse Chebyshev low pass filter. In this work, a program was developed in Matlab working environment [20,21]; the steps are presented in Table 3.

Table 3 Program to make the pole-zero plot, obtain the transfer function, and plot the frequency response of Inverse Chebyshev low pass filter

Step	Description
1	Input A_{pass}, A_{stop}, ω_{pass}, and ω_{stop}
2	Calculate ε_i, n, and D_i using Eqns. (43), (44), and (47) respectively; n is rounded up to the next higher whole number using the code "ceil" in Matlab [20].

3	Plot the Chebyshev ellipse with major axis $\pm \cosh(D_i)$ along the jω axis and minor axis $\pm \sinh(D_i)$ along the real axis on the s-plane. For each point on the Chebyshev ellipse, find the reciprocal (Eqn. (46)) and plot the Inverse Chebyshev ellipse on the same s-plane using a different colour. The following sub-program is suggested.
	 nn1=0;
	 for t=0:2*pi/10000:2*pi
	  nn1=nn1+1;
	  x(nn1)=-sinh(Di)*sin(t);
	  y(nn1)=cosh(Di)*cos(t);
	  yy1(nn1)=-y(nn1)/((x(nn1))^2+(y(nn1))^2);
	  xx1(nn1)=x(nn1)/((x(nn1))^2+(y(nn1))^2);
	 end
	 plot(x,y,'g--')
	 hold
	 plot(xx1,yy1,'m--')

4	If n is odd, get the real pole at $\sigma = -sinh(D)$. Plot the Chebyshev real pole on the s-plane. Plot its reciprocal as Inverse Chebyshev real pole.
5	Set m=-1.
6	Increase m by 1.
7	Calculate θ_m, using Eqn. (50) or (51). Calculate and plot Chebyshev pole σ'_m and w'_m using Eqns. (48) and (49). Obtain the reciprocal pole (the desired Inverse Chebyshev pole) σ_m, w_m, B_{1m}, and B_{2m} using Eqns. (52), (53), (58), and (59) respectively. Plot the Chebyshev pole and its conjugate on the s-plane. Also plot the Inverse Chebyshev pole and its conjugate on the s-plane. Obtain the Inverse Chebyshev zero σ_{zm}, ω_{zm}, A_{1m}, and A_{2m} using Eqns. (54), (55), (60), and (61) respectively. Plot the Inverse Chebyshev zero and its conjugate on the same s-plane.
8	Is $m < (n/2) - 1$ for even order or is $m < [(n-1)/2] - 1$ for odd order? If Yes, go back to Step 6. If No, proceed to Step 9.
9	Obtain the transfer function using Eqn. (62) or (63).
10	Plot the frequency response

2.4. Elliptic Approximation

The elliptic or Cauer approximation function provides the best selectivity characteristic compared with other approximation functions [17]. No other approximation method will be able to provide a lower-order filter for any given set of specifications. The elliptic filter combines ripple in the passband and stopband in order to accomplish this feat. There are elliptic functions that have no trigonometric counterparts [23].

The elliptic approximation's magnitude frequency response function is given by Eqn. (63). R_n is the Chebyshev rational function of order n. R_n is composed of both numerator and denominator portions, which allow an equiripple response in both the passband and stopband. The Chebyshev rational function R_n and much of elliptic approximation theory is based on the elliptic integral and the Jacobian elliptic functions.

$$|H_{E,n}[j(w/w_0)]| = \frac{1}{\sqrt{1+\varepsilon^2 R_n^2(w_0/w)}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (63)$$

The order of the elliptic approximation function required to meet the specifications for a filter is given as

$$n_E = \frac{CE1(rt)CEI(\sqrt{1-kn^2})}{CEI(\sqrt{1-rt^2})CEI(kn)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (64)$$

where the ratio rt and the kernel kn are defined as

$$rt = w_{pass}/w_{stop} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (65)$$

$$kn = \sqrt{(10^{-0.1A_{pass}} - 1)/(10^{-0.1A_{stop}} - 1)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (66)$$

CEI refers to the Complete Elliptic Integral. CEI is defined as in Eqn. (67). Numerical Integration is used to evaluate the integral [24]. The area under the curve of f(x) between $x = 0$ and $x = \phi$ is divided into small elements of width Δx as illustrated in Fig. 7 [24]. The sum of the areas of the elements tends to be equal to the integral as Δx approaches 0 as described by Eqn. (68).

$$u(\phi, k) = \int_0^\phi \sqrt{(1 - k^2 sin^2 x)}\, dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (67)$$

$$u(\phi, k) = \int_0^\phi \sqrt{(1 - k^2 sin^2 x)}\, dx = \int_0^\phi f(x)dx = \lim_{\Delta x \to 0}(f_1 + f_2 + f_3 + \ldots + f_N)\Delta x \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (68)$$
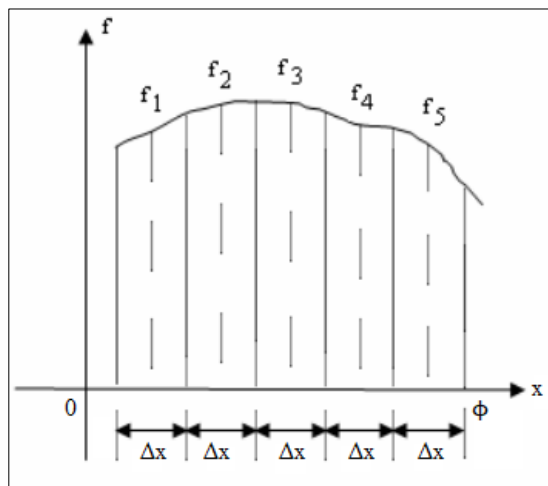

Figure 7 Numerical integration [24]

To locate the poles and zeros of the Elliptic approximation filter, this integral CEI is used in two approaches. The forward approach is used to determine CEI (k). In this case take ϕ to be $\pi/2$, integrate between the lower limit 0 and upper limit $\pi/2$. A subprogram was developed to perform this numerical integration as presented in Table 4. To evaluate $CE1(rt)$ in Eqn. 64, run the CEI forward approach subprogram of Table 4 with $k = rt$. To evaluate $CEI(\sqrt{1 - kn^2})$ in Eqn. 64, run

the CEI forward approach subprogram of Table 4 with $k = \sqrt{1 - kn^2}$. What value of k should be used to evaluate $CEI(\sqrt{1 - rt^2}$? What value of k should be used to evaluate $CEI(\text{kn})$?

Table 4 Subprogram for numerical integration of CEI (forward approach)

Step	Description
1	Input k. Set phai (ϕ) to π/2.

2	sum=0;
	for x=0:phai/1000000:phai
	 sum=sum+((1-k^2*(sin(x))^2)^-0.5)*phai/1000000;
	end

3	Output sum as CEI(k)

The reverse approach is used to handle the quantities in Eqns. (69) to (73). Here, the value of the integral CEI is given as u and the constant k is also given, the value of ϕ the upper limit of the integral is required before finding its sine or cosine or tangent as the case may be. A subprogram was developed to perform this numerical integration as presented in Table 5. The required ϕ is the value of x that gives the value of the integral as u.

$$sn(u, k) = sin(\phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (69)$$

$$cn(u, k) = cos(\phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (70)$$

$$sc(u, k) = tan(\phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (71)$$

$$sc^{-1}(u, k) = tan^{-1}(\phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (72)$$

$$dn(u, k) = \frac{d\phi}{dx} = \sqrt{(1 - k^2 sn^2(u, k)} = \sqrt{(1 - k^2 sin^2(\phi)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (73)$$

Table 5 Subprogram for numerical integration of CEI (reverse approach)

Step	Description
1	Input u and k.

2	sum=0;
	cnt=0;
	for x=0:1/100000:5*pi
	 sum=sum+((1-k^2*(sin(x))^2)^-0.5)*1/100000;
	 cnt=cnt+1;
	if sum==u
	break
	elseif sum>u
	break
	end
	End
	phai=x;

3	Output ϕ. Compute its sine, cosine, tangent or tan^{-1} as may be required.

A variable v_0 which is used in the calculation of the pole and zero locationsis defined as

$$v_0 = \frac{CEI(rt).sc^{-1}(\varepsilon^{-1},kn)}{n.CEI(kn)} \qquad \text{......................} \qquad (74)$$

Next, the pole's real and imaginary components are determined as

$$\sigma_m = -\frac{cn[f(m),rt].dn[f(m),rt].sn(v_0,\sqrt{1-rt^2}).cn(v_0,\sqrt{1-rt^2})}{1-dn^2[f(m),rt].sn^2(v_0,\sqrt{1-rt^2})} \qquad \text{......................} \qquad (75)$$

$$w_n = \frac{sn[f(m),rt].dn(v_0,\sqrt{1-rt^2})}{1-dn^2[f(m),rt].sn^2(v_0,\sqrt{1-rt^2})} \qquad \text{......................} \qquad (76)$$

where

$$f(m) = \frac{CEI(rt)(2m+1)}{n}, m = 0,1,\dots,(n/2)-1 \text{ (n even)} \qquad \text{......................} \qquad (77)$$

$$f(m) = \frac{CEI(rt)(2m+2)}{n}, m = 0,1,\dots,[(n-1)/2]-1 (n\ odd) \qquad \text{......................} \qquad (78)$$

Note the negative sign for σ_m, which effectively moves the pole location from the first quadrant to the second quadrant. For odd-order functions, there is a pole on the real axis at

$$\sigma_R = -\frac{sn(v_0,\sqrt{1-rt^2}).cn(v_0,\sqrt{1-rt^2}))}{1-sn^2(v_0,\sqrt{1-rt^2})} \qquad \text{......................} \qquad (79)$$

The location of the zeros that will be purely imaginary on the jω axis are given by

$$\sigma_{zm} = 0.0 \qquad \text{......................} \qquad (80)$$

$$w_{zm} = \frac{1}{rt.sn[f(m),rt]} \qquad \text{......................} \qquad (81)$$

To evaluate $sc^{-1}(\varepsilon^{-1}, kn)$ in Eqn. (74), run the subprogram of the reverse approach in Table 5 with $k = kn$ and $u = \varepsilon^{-1}$ to get ϕ, then compute $tan^{-1}(\phi)$. To evaluate $sn(v_0, \sqrt{1-rt^2})$ in Eqn. (75), run the CEI reverse approach subprogram of Table 5 with $k = \sqrt{1-rt^2}$ and $u = v_0$ to get ϕ, then compute $sin(\phi)$. Square the value obtained for $sn(v_0, \sqrt{1-rt^2})$ to get $sn^2(v_0, \sqrt{1-rt^2})$. To evaluate $dn[f(m),rt]$ in Eqn. (75), run the CEI reverse approach subprogram of Table 5 with $k = rt$ and $u = f(m)$ to get ϕ, then compute $\sqrt{(1-k^2 sin^2(\phi)}$ in accordance with Eqn. (73). Square the value obtained for $dn[f(m),rt]$ to get $dn^2[f(m),rt]$ in Eqn. (76).

The complete Elliptic approximation transfer function can be determined from a combination of a first-order factor (for odd orders) presented in Eqn. (82) and quadratic factors presented in Eqn. (83) and explained in Eqns. (84) to (87). Each of these pole factors will have a constant in the numerator and each of these zero factors will have a constant in the denominator to adjust the Gain to unity (0 dB) at $\omega = 0$. The complete Elliptic transfer function can now be defined as shown in Eqns. (88) and (89) for even-order and odd-order respectively. Furthermore, a Gain constant G given by Eqn. (39) must be included for even-order transfer functions to adjust the Gain to unity (0 dB) at $\omega = 0$.

As a class project, students should be tasked to develop a computer program to make the pole-zero plot, obtain the transfer function, and plot the frequency response for the Elliptic low pass filter. In this work, a program was developed in Matlab working environment [20,21]; the steps are presented in Table 6.

$$H_0(S) = \frac{\sigma_R}{S+\sigma_R} \qquad \text{......................} \qquad (82)$$

$$H_m(s) = \frac{B_{2m}(S^2+A_{1m}S+A_{2m})}{A_{2m}(S^2+B_{1m}S+B_{2m})} \qquad \text{......................} \qquad (83)$$

$$B_{1m} = -2\sigma_m \qquad \text{......................} \qquad (84)$$

$$B_{2m} = \sigma_m^2 + w_m^2 \qquad \text{......................} \qquad (85)$$

Table 6 Program to make the pole-zero plot, obtain the transfer function, and plot the frequency response of Elliptic low pass filter

Step	Description
1	Input A_{pass}, A_{stop}, ω_{pass}, and ω_{stop}
2	Calculate ε, rt, kn, $\sqrt{1-kn^2}$, $\sqrt{1-rt^2}$; CEI(rt), CEI(kn), $CEI(\sqrt{1-rt^2})$, $CEI(\sqrt{1-Kn^2})$, and n; n is rounded up to the next higher whole number using the code "ceil" in Matlab [20].

3	u=ε^-1;
	k=kn;
	Run the Subprogram Table 5
	x1=x;
	vo=CEI(rt)*atan(x1)/(n*CEI(kn));
	u=vo;
	k=$\sqrt{1-rt^2}$;
	Run Subprogram Table 5
	x2=x;

4	Plot the ellipse on the s-plane. m is varied from 0, 0.1, 0.2, 0.3, …, to 2n . The upper limit of 2n was obtained by trial and error.
	nn=0;
	figure
	for m=0:0.1:2*n
	 nn=nn+1;
	 f(nn)=CEI(rt)*(2*m+2)/n;
	u=f(nn);
	 k=rt;
	Run the Subprogram Table 5
	 x3=x;
	 xx1(nn)=-cos(x3)*(1-(rt*sin(x3))^2)^0.5*sin(x2)*cos(x2)/(1-(1-(rt*sin(x3))^2)*(sin(x2))^2);
	 yy1(nn)=sin(x3)*(1-(rtt*sin(x2))^2)^0.5/(1-(1-(rt*sin(x3))^2)*(sin(x2))^2);
	end
	plot(xx1,yy1,'m--')

5	If n is odd, get the real pole at σ_R from Eqn. (79). Plot the Elliptic real pole on the s-plane. If n is even, obtain G from Eqn. (39).
6	Set m=-1.
7	Increase m by 1.
8	Calculate f_m, using Eqns. (77) or (78). Compute v_0 using Eqn. (74) Calculate the Elliptic pole σ_m, w_m, B_{1m}, and B_{2m} using Eqns. (75), (76), (84), and (85) respectively. Plot the Elliptic pole and its conjugate on the s-plane. Obtain the Elliptic zero σ_{zm}, ω_{zm}, A_{1m}, and A_{2m} using Eqns. (80), (81), (86), and (87) respectively. Plot the Elliptic zero and its conjugate on the s-plane.
9	Is $m < (n/2)-1$ for even order or is $m < [(n-1)/2]-1$ for odd order? If Yes, go back to Step 7. If No, proceed to Step 10.
10	Obtain the transfer function.
11	Plot the frequency response

$$A_{1m} = -2\sigma_{zm} = 0.0 \quad \text{.....................} \quad (86)$$

$$A_{2m} = \sigma_{zm}^2 + w_{zm}^2 = w_{zm}^2 \quad \text{.....................} \quad (87)$$

$$H_{E,n}(S) = \frac{G \prod_{m=0}^{(n/2)-1}(B_{2m}) \prod_{m=0}^{(n/2)-1}(S^2 + A_{1m}S + A_{2m})}{\prod_{m=0}^{(n/2)-1}(A_{2m}) \prod_{m=0}^{(n/2)-1}(S^2 + B_{1m}S + B_{2m})}, \text{(n even)} \quad \text{.....................} \quad (88)$$

$$H_{E,n}(S) = \frac{\sigma_R \prod_{m=0}^{[(n-1)/2]-1}(B_{2m}) \prod_{m=0}^{[(n-1)/2]-1}(S^2 + A_{1m}S + A_{2m})}{(S+\sigma_R) \prod_{m=0}^{[(n-1)/2]-1}(A_{2m}) \prod_{m=0}^{[(n-1)/2]-1}(S^2 + B_{1m}S + B_{2m})}, \text{(n odd)} \quad \text{.....................} \quad (89)$$

2.5. Bessel Approximation

The Bessel filter is optimized to obtain better transient response due to a linear phase (constant delay) in the passband. This means that there will be relatively poorer frequency response (less amplitude discrimination) [25]. Bessel filter has a wider transition band, but its phase is linear within the passband [26]. Bessel filters are a class of all-pole filters [27]. The transfer function is given as

$$H(s) = \frac{B_0}{B_n(S)} \quad \text{.....................} \quad (90)$$

where $B_n(S)$ is the nth order Bessel polynomial.

This polynomial can be expressed as in Eqn. (91) [27].

$$B_n(S) = \sum_{k=0}^{n} a_k s^k \quad \text{.....................} \quad (91)$$

where the coefficients a_k are given as

$$a_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!}, \quad k = 0,1,\dots,n \quad \text{.....................} \quad (92)$$

The coefficients are obtainable by a simple program. The transfer function of Eqn. (90) has a constant B_0 in the numerator to adjust the Gain to unity (0 dB) at $\omega = 0$. The poles are obtainable by finding the roots of $B_n(s)$ using existing methods in [20] or [28].

3. Results and Discussions

Programs were developed for the five approximation functions. The programs were subjected to tests. The results for two sets of low pass filter specifications for each approximation function are presented and discussed in this section.

3.1. Butterworth Approximation

The first set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -20$ dB. The results are presented in Table 7 and Fig. 8. In the pole-zero plot of Fig. 8(a), the poles are represented with x in colour blue and the conjugate poles are represented with x in colour red. Other corresponding points which have equal but opposite real part with the poles are represented with * in colour black. These points represented with * are not poles and they are in the right half of the s-plane. These representations are adopted in other pole-zero plots in this section. The response in Fig. 8(b) is flat in the passband 0 to 1 rad/sec. At the stopband edge frequency of 2 rad/sec, the Gain is less than -20 dB as specified.
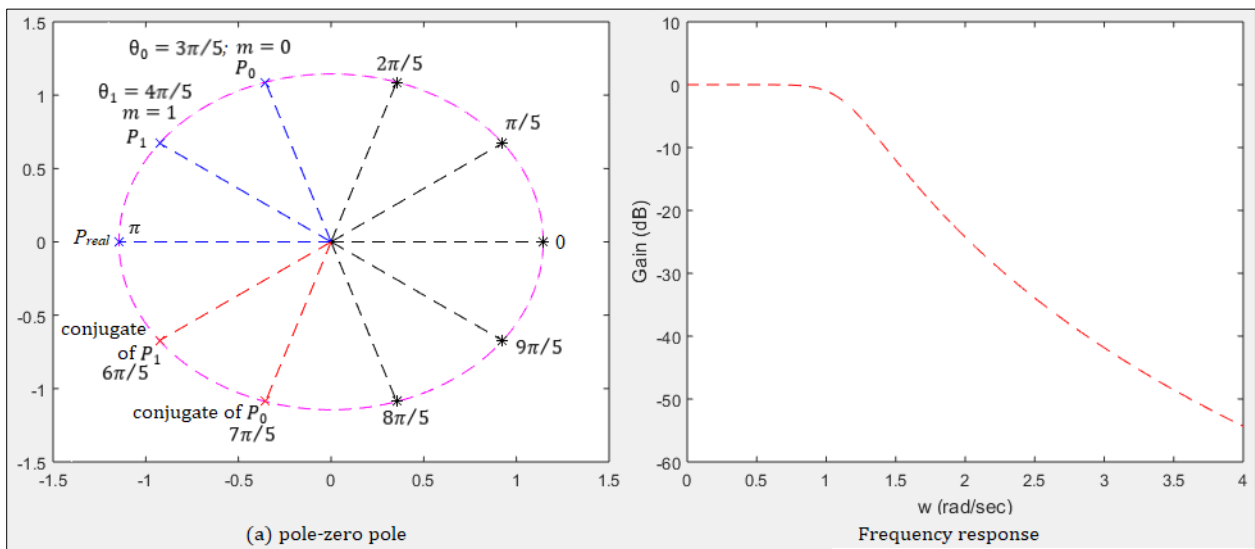
The second set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -25$ dB. The results are presented in Table 8 and Fig. 9. The response in Fig. 9(b) is flat in the passband 0 to 1 rad/sec. At the stopband edge frequency of 2 rad/sec, the Gain is less than -25 dB as specified.

Generally for Butterworth approximation, the locus of the poles is a circle. Lines linking the poles with the origin are separated by equal angles.
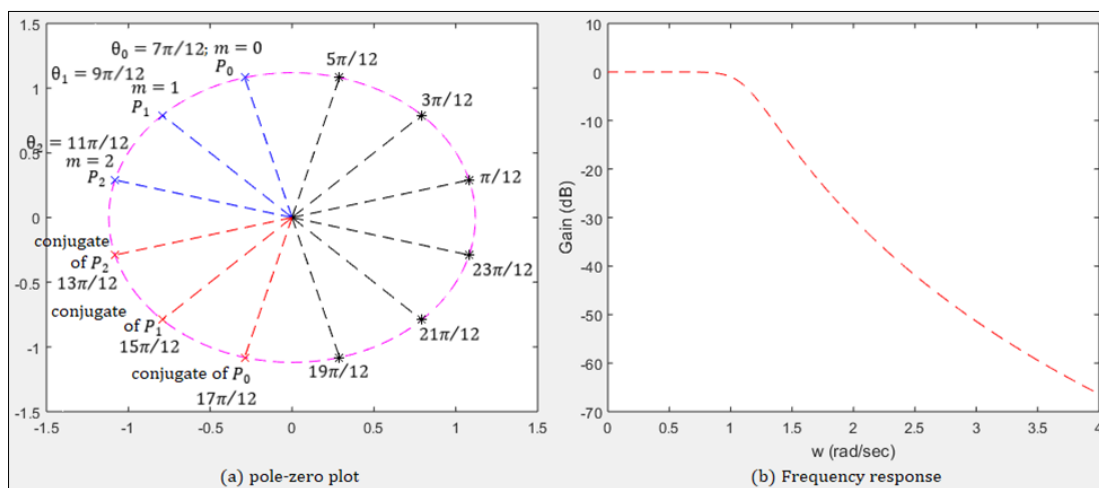
Table 7 Parameters, poles, and transfer function of Butterworth low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -20$ dB

$\varepsilon = 0.5088$	$R = 1.1447$	$n = 5$
$P_{real} = -1.1447, P_0 = -0.3537 + j1.0887, P_1 = -0.9261 + j0.6728$		
$H(s) = \dfrac{(1.1447)(1.3103)(1.3103)}{(s + 1.1447)(s^2 + 0.7074s + 1.3103)(s^2 + 1.8521s + 1.3103)}$		
$H(s) = \dfrac{1.9652}{s^5 + 3.7042s^4 + 6.8607s^3 + 7.8533s^2 + 5.5558s + 1.9652}$		

Figure 8 Pole-zero plot and frequency response for Butterworth low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -20$ dB [n obtained as 5]
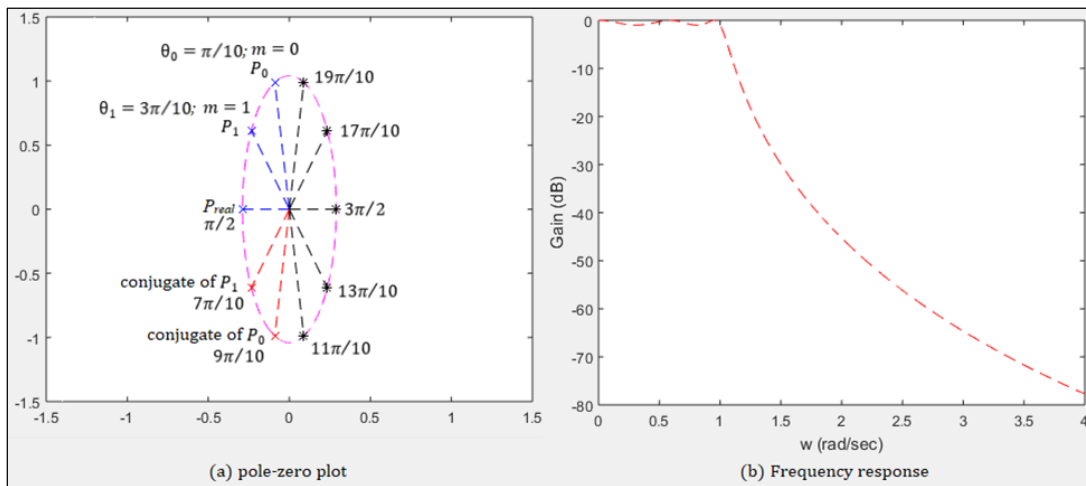

Figure 9 Pole-zero plot and frequency response for Butterworth low pass filter with the second set of specifications:$w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -25$ dB [n obtained as 6]

Table 8 Parameters, poles, and transfer function of Butterworth low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -25$ dB

$\varepsilon = 0.5088$	$R = 1.1192$	$n = 6$
$P_0 = -0.2897 + j1.0811, P_1 = -0.7914 + j0.7914, P_2 = -1.0811 + j0.2897$		
$H(s) = \dfrac{(1.2526)(1.2526)(1.2526)}{(s^2 + 0.5793s + 1.2526)(s^2 + 1.5828s + 1.2526)(s^2 + 2.1621s + 1.2526)}$		
$H(s) = \dfrac{1.9652}{s^6 + 4.3242s^5 + 9.3494s^4 + 12.8153s^3 + 11.7108s^2 + 6.7844s + 1.9652}$		

3.2. Chebyshev approximation

The first set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -40$ dB. The results are presented in Table 9 and Fig. 10. The response in Fig. 10(b) shows ripples in the passband 0 to 1 rad/sec. At the stopband edge frequency of 2 rad/sec, the Gain is less than -40 dB as specified.

Table 9 Parameters, poles, and transfer function of Chebyshev low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -40$ dB

| $\varepsilon = 0.5088$	$D = 0.2895$	$n = 5$
$P_{real} = -0.2895, P_0 = -0.0895 + j0.9901, P_1 = -0.2342 + j0.6119$		
$H(s) = \dfrac{(0.2895)(0.9883)(0.4293)}{(s + 0.2895)(s^2 + 0.1789s + 0.9883)(s^2 + 0.4684s + 0.4293)}$		
$H(s) = \dfrac{0.1228}{s^5 + 0.9368s^4 + 1.6888s^3 + 0.9744s^2 + 0.5805s + 0.1228}$		

Figure 10 Pole-zero plot and frequency response for Chebyshev low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -40$ dB [n obtained as 5]

The second set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -50$ dB. The results are presented in Table 10 and Fig. 11. The response in Fig. 11(b) shows ripples in the passband 0 to 1 rad/sec. At the stopband edge frequency of 2 rad/sec, the Gain is less than -50 dB as specified.

Generally for Chebyshev approximation, the locus of the poles is an ellipse. Lines linking the poles with the origin are separated by equal angles.

Table 10 Parameters, poles, and transfer function of Chebyshev low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -50$ dB

$\varepsilon = 0.5088$	$D = 0.2380$	$n = 6$
$P_0 = -0.0622 + j0.9934, P_1 = -0.1699 + j0.7272, P_2 = -0.2321 + j0.2662$		
$H(s) = \dfrac{(0.8913)(0.9907)(0.5577)(0.1247)}{(s^2 + 0.1244s + 0.9907)(s^2 + 0.3398s + 0.5577)(s^2 + 0.4641s + 0.1247)}$		
$H(s) = \dfrac{0.0614}{s^6 + 0.9283s^5 + 1.9308s^4 + 1.2021s^3 + 0.9393s^2 + 0.3071s + 0.0689}$		

(a) pole-zero plot (b) Frequency response
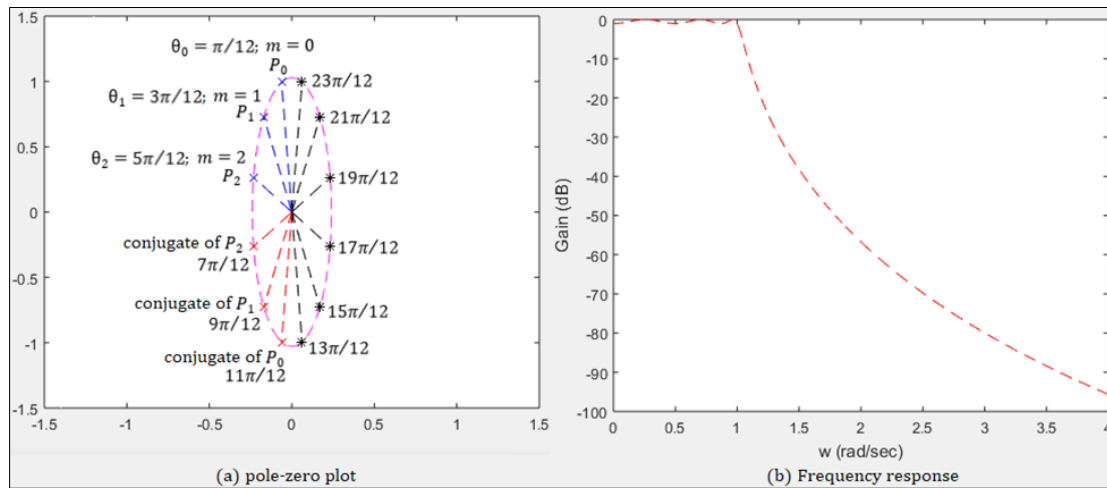
Figure 11 Pole-zero plot and frequency response for Chebyshev low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -50$ dB [n obtained as 6]

3.3. Inverse Chebyshev approximation

The first set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -40$ dB. The results are presented in Table 11 and Fig. 12. The zeros are represented with o in colour blue. The zeros occur in conjugate pairs. The response in Fig. 12(b) is flat in the passband 0 to 1 rad/sec and there are ripples in the stopband. At the stopband edge frequency of 2 rad/sec, the Gain is less than -40 dB as specified.

(a) pole-zero plot (b) Frequency response
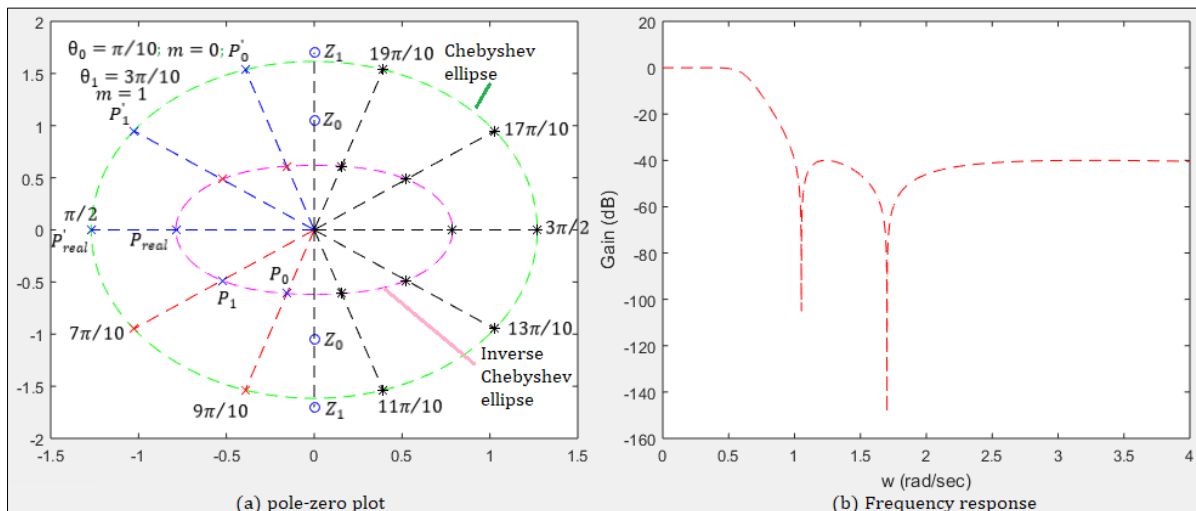
Figure 12 Pole-zero plot and frequency response for Inverse Chebyshev low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -40$ dB [n obtained as 5]
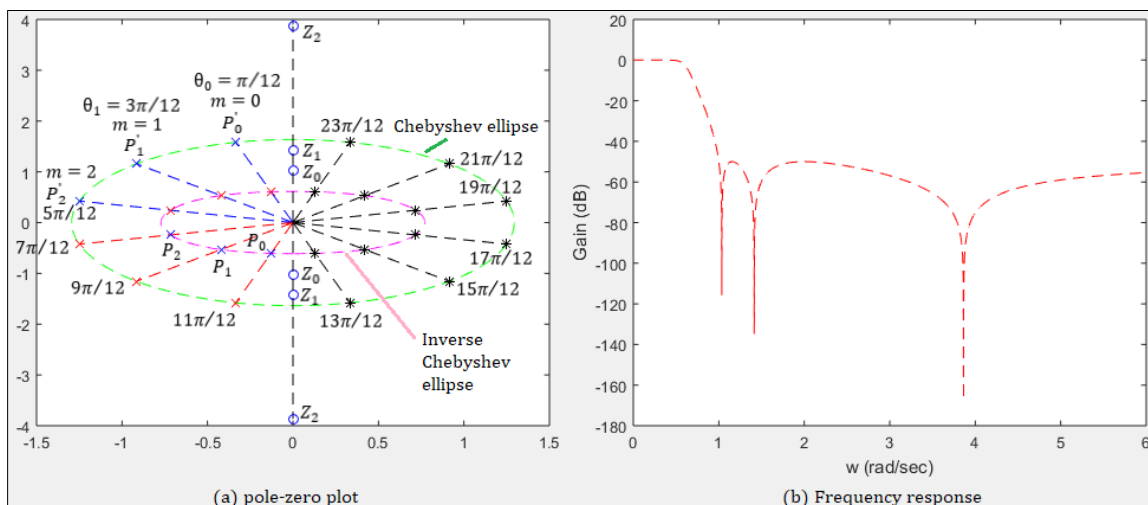
Table 11 Parameters, poles, zeros, and transfer function of Inverse Chebyshev low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -40$ dB

$\varepsilon_i = 0.010$	$D_i = 1.0597$	$n = 5$
$P_{real} = -0.7878, P_0 = -0.1559 - j0.6109, P_1 = -0.5248 - j0.48541$		
$Z_0 = 0 + j1.0515, Z_1 = 0 + j1.7013$		
$H(s) = \dfrac{(0.7878)(0.3975)(0.5110)(s^2 + 1.1056)(s^2 + 2.8944)}{(1.1056)(2.8944)(s + 0.7878)(s^2 + 0.3118s + 0.3975)(s^2 + 1.0496s + 0.5110)}$		
$H(s) = \dfrac{0.05(s^4 + 4s^2 + 3.2)}{s^5 + 2.1492s^4 + 2.3083s^3 + 1.5501s^2 + 0.6573s + 0.16}$		

The second set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -50$ dB. The results are presented in Table 12 and Fig. 13. The response in Fig. 13(b) is flat in the passband 0 to 1 rad/sec and there are ripples in the stopband. At the stopband edge frequency of 2 rad/sec, the Gain is less than -50 dB as specified.

Table 12 Parameters, poles, zeros, and transfer function of Inverse Chebyshev low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -50$ dB

$\varepsilon_i = 0.0032$	$D_i = 1.0749$	$n = 6$
$P_0 = -0.1284 - j0.6057, P_1 = -0.4208 - j0.5317, P_2 = -0.7176 - j0.2430$		
$Z_0 = 0 + j1.0353, Z_1 = 0 + j1.4142, Z_2 = 0 + j3.8637$		
$H(s) = \dfrac{(0.3834)(0.4598)(0.5740)(s^2 + 1.0718)(s^2 + 2)(s^2 + 14.9282)}{A_0A_1A_2(s^2 + 0.2569s + 0.3834)(s^2 + 0.8415s + 0.4598)(s^2 + 1.4353s + 0.5740)}$ where $A_0A_1A_2 = (1.0718)(2)(14.9282)$		
$H(s) = \dfrac{3.1625 \, x10^{-3}(s^6 + 18s^4 + 48s^2 + 32)}{s^6 + 2.5336s^5 + 3.2098s^4 + 2.5917s^3 + 1.417s^2 + 0.5060s + 0.1012}$		

Figure 13 Pole-zero plot and frequency response for Inverse Chebyshev low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -50$ dB [n obtained as 6]

Generally for Inverse Chebyshev approximation, the locus of the poles is an ellipse. There are two ellipses. Chebyshev poles $P_0', P_1', P_2', \ldots$ are on the Chebyshev ellipse. There corresponding reciprocals $P_0, P_1, P_2, \ldots$ are the actual Inverse Chebyshev poles which are on the Inverse Chebyshev ellipse. Lines linking the Inverse Chebyshev poles with the origin

are separated by equal angles. The number of minimums (turning points) in the response in the stopband increases with n; there are two turning points for $n = 5$ as shown in Fig. 12(b) but there are three turning points for $n = 6$ as shown in Fig. 13(b).

3.4. Elliptic approximation

The first set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -60$ dB. The results are presented in Table 13 and Fig. 14. There are zeros in conjugate pairs. There are ripples in both the passband and stopband as shown in Fig. 14(b). At the stopband edge frequency of 2 rad/sec, the Gain is less than -60 dB as specified.

Table 13 Parameters, poles, zeros, and transfer function of Elliptic low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -60$ dB

| $\varepsilon = 0.5088$	$rt = 0.5$	$kn = 5.0885 \, x10^{-4}$	$v_0 = 0.2361$	$n = 5$
$P_{real} = -0.2389, P_0 = -0.1782 + j0.6336, P_1 = -0.0599 + j0.9784$				
$Z_0 = 0 + j3.2508, Z_1 = 0 + j2.0892$				
$H(s) = \dfrac{(0.2389)(0.4332)(0.9609)(s^2 + 10.5677)(s^2 + 4.3650)}{(10.5677)(4.3650)(s + 0.2389)(s^2 + 0.3565s + 0.4332)(s^2 + 0.1197s + 0.9609)}$				
$H(s) = \dfrac{2.1549 \, x10^{-3}(s^4 + 14.9327s^2 + 46.1278)}{s^5 + 0.7151s^4 + 1.5506s^3 + 0.7376s^2 + 0.5105s + 0.0994}$				

(a) pole-zero plot   (b) Frequency response
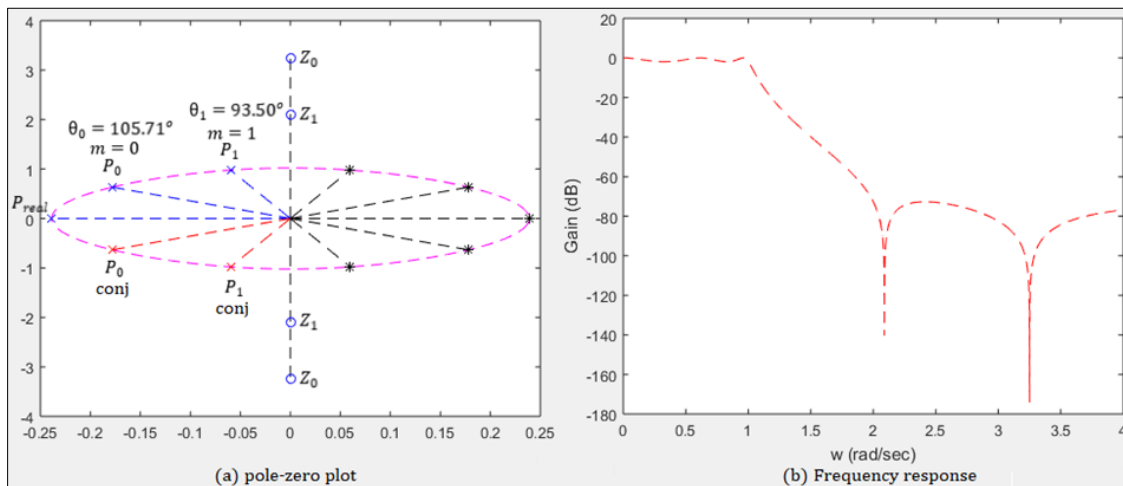
Figure 14 Pole-zero plot and frequency response for Elliptic low pass filter with the first set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -60$ dB [n obtained as 5]

The second set of specifications is $w_{pass} = 1$ rad/sec, $w_{stop} = 2$ rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -70$ dB. The results are presented in Table 14 and Fig. 15. At the stopband edge frequency of 2 rad/sec, the Gain is less than -70 dB as specified.

Generally for Elliptic approximation, the locus of the poles is an ellipse. Lines linking the Elliptic poles with the origin are separated by non-equal angles. The number of minimums (turning points) in the response in the stopband increases with n; there are two for $n = 5$ as shown in Fig. 14(b) but there are three turning points for $n = 6$ as shown in Fig. 15(b).

Table 14 Parameters, poles and transfer function of Elliptic low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -70$ dB

$\varepsilon = 0.5088$	$rt = 0.5$	$kn = 1.6091 \, x10^{-4}$	$v_0 = 0.1968$	$n = 6$
$P_0 = -0.1887 + j0.2829, P_1 = -0.1251 + j0.7460, P_2 = -0.0415 + j0.9850$				

$$Z_0 = 0 + j7.2360, Z_1 = 0 + j2.7321, Z_2 = 0 + j2.0611$$

$$H(s) = \frac{(0.1157)(0.5722)(0.9719)(0.8913)(s^2 + 52.3603)(s^2 + 7.4642)(s^2 + 4.2482)}{A_0 A_1 A_2 (s^2 + 0.3773s + 0.1157)(s^2 + 0.2502s + 0.5722)(s^2 + 0.0831s + 0.9719)}$$

Where $A_0 A_1 A_2 = (52.3603)(7.4642)(4.2482)$

$$H(s) = \frac{3.4512 \, x10^{-5}(s^6 + 64.1s^4 + 645s^2 + 1660.3)}{s^6 + 0.7106s^5 + 1.8063s^4 + 0.9197s^3 + 0.8468s^2 + 0.2435s + 0.0643}$$
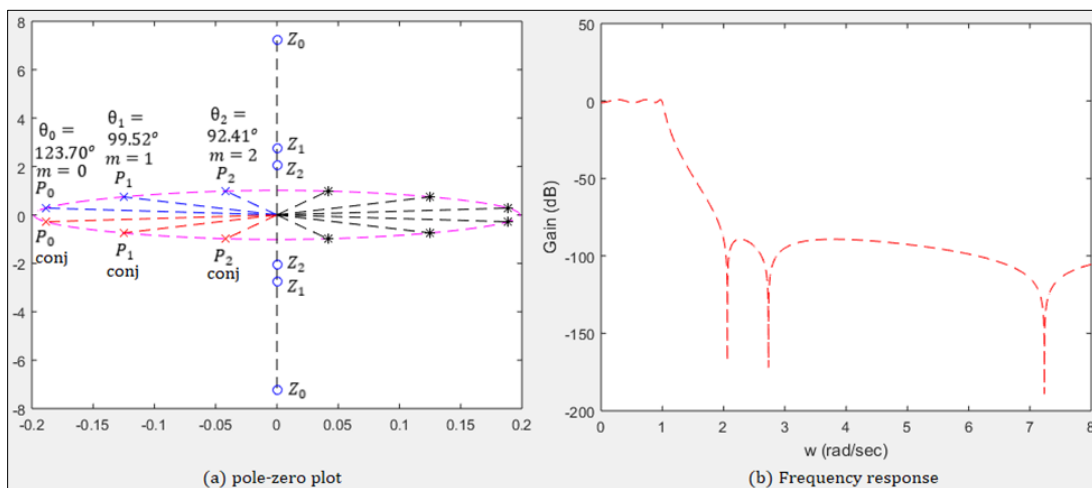
Figure 15 Pole-zero plot and frequency response for Elliptic low pass filter with the second set of specifications: $w_{pass} = 1$ rad/sec, $w_{stop} = 2$rad/sec, $A_{pass} = -1$ dB, and $A_{stop} = -70$ dB [n obtained as 6]

3.5. Bessel approximation

Bessel approximation function is different from the others as there is no formulae relating the order number n with w_{pass}, w_{stop}, A_{pass}, and A_{stop}. $n = 5$ was used as the first input. The results are presented in Table 15 and Fig. 16. There are no zeros. Values of the Gain for certain frequencies are also tabulated in Table 15. There is a wider transition bandwidth. Passband Gain is -0.4865 dB at passband edge frequency of 1 rad/sec. A reasonable stopband Gain of -41.2426 dB is achievable at 10 rad/sec; a decade wide transition band.

Table 15 Poles and transfer function of Bessel low pass filter with $n = 5$

w	0	1	2	5	10	20	rad/sec
A	0	-0.4865	-2.0012	-14.9409	-41.2426	-70.7628	dB
$P_{real} = -3.6467, P_0 = -2.3247 + j3.5710, P_1 = -3.3520 + j1.7427$							
$H(s) = \dfrac{945}{s^5 + 15s^4 + 105s^3 + 420s^2 + 945s + 945}$							

$n = 6$ is used as the second input. The results are presented in Table 16 and Fig. 17. Passband Gain is -0.3968 dB at passband edge frequency of 1 rad/sec. A reasonable Stopband Gain of -40.7959 dB is achievable at 10 rad/sec; a decade wide transition band.
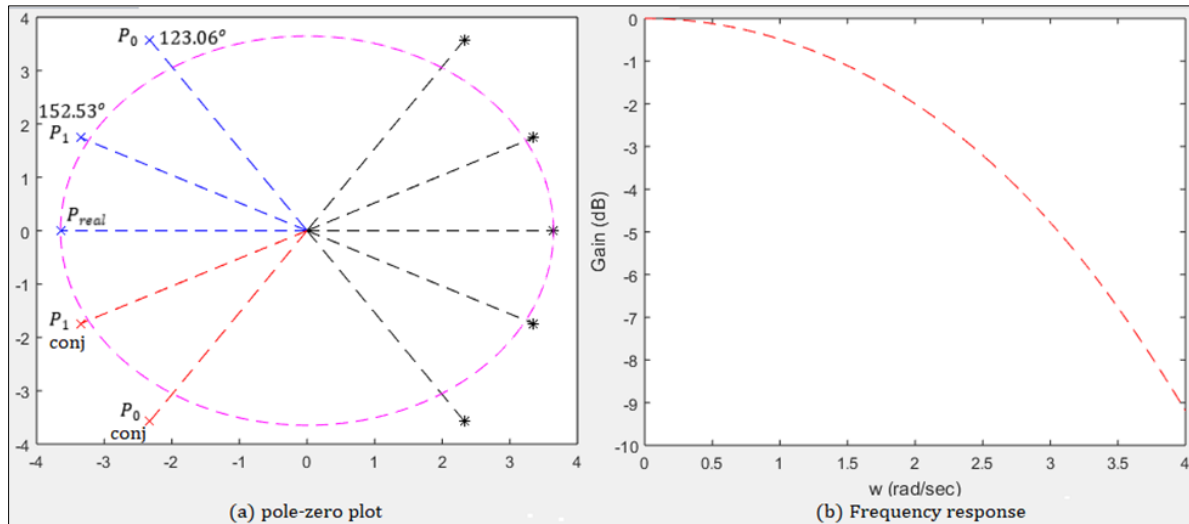
Figure 16 Pole-zero plot and frequency response for Bessel low pass filter with $n = 5$

Table 16 Parameters, poles and transfer function of Bessel low pass filter with $n = 6$

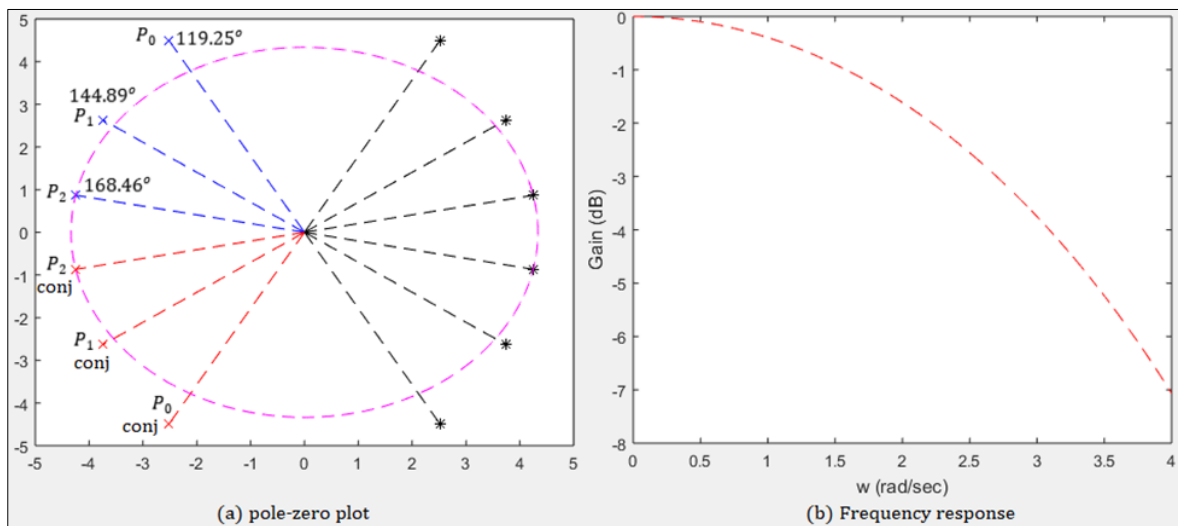| w	0	1	2	5	10	20	rad/sec
A	0	-0.3968	-1.6137	-11.8495	-40.7959	-76.0266	dB
$P_0 = -2.5159 + j4.4927, P_1 = -3.7357 + j2.6263, P_2 = -4.2484 + j0.8675$							
$H(s) = \dfrac{10395}{s^6 + 21s^5 + 210s^4 + 1260s^3 + 4725s^2 + 10395s + 10395}$							

Figure 17 Pole-zero plot and frequency response for Bessel low pass filter with with $n = 6$

Generally for Bessel approximation, the locus of the poles is neither a circle nor an ellipse. Lines linking the Bessel poles with the origin are separated by non-equal angles.

3.6. Comparison of approximation functions

Bessel function approximation was further investigated by studying the values of the Gain over the frequency range 0 to 100 rad/sec for order 1, 2, 3, 4, 5, 6, 7, and 12 as presented in Table 17. The trend in Table 17 confirmed the decade wide transition band characteristics of Bessel function approximation. Generally, reasonable stopband Gain is

achievable at a stopband edge frequency of 10 rad/sec; a decade away from the passband edge frequency of 1 rad/sec. Linear-phase response over the passband is a characteristic of Bessel function approximation but suffers poor attenuation in the stopband.

Table 17 Results of further investigation of Bessel approximation function

w(rad/sec)	0	1	2	5	10	20	50	100
order (n)	Gain (dB)							
1	0	-3.01	-6.99	-14.15	-20.04	-26.03	-33.98	-40.00
2	0	-1.60	-6.14	-18.96	-30.59	-42.53	-58.42	-70.46
3	0	-0.90	-4.00	-19.64	-36.75	-54.61	-78.43	-96.48
4	0	-0.63	-2.67	-17.91	-40.05	-63.77	-95.51	-119.58
5	0	-0.49	-2.00	-14.94	-41.24	-70.76	-110.41	-140.50
6	0	-0.40	-1.61	-11.85	-40.80	-76.03	-123.58	-159.67
7	0	-0.34	-1.36	-9.46	-39.05	-79.85	-135.29	-177.40
12	0	-0.19	-0.76	-4.85	-22.09	-83.31	-177.89	-250.03

The response of the other four approximation functions for the same set of specifications are recorded and compared in Table 18. Observation of the trends in Table 18 confirmed the following facts. For any set of specifications, Elliptic approximation requires less order number while Butterworth approximation requires highest order number; Chebyshev and Inverse Chebyshev approximations have the same order number requirement which is usually greater than that of Elliptic approximation but less than that of Butterworth approximation. Higher w_{pass}/w_{stop} ratio which means wider transition band requires less order number. Higher negative stopband Gain requires higher order number.

Table 18 Comparison of four approximation functions for the same set of specifications

	$w_{pass} = 1$ rad/sec $w_{stop} = 2$rad/sec $w_{stop}/w_{pass} = 2$ $A_{pass} = -3$ dB $A_{stop} = -70$ dB	$w_{pass} = 1$ rad/sec $w_{stop} = 4$rad/sec $w_{stop}/w_{pass} = 4$ $A_{pass} = -3$ dB $A_{stop} = -70$ dB	$w_{pass} = 1$ rad/sec $w_{stop} = 3$rad/sec $w_{stop}/w_{pass} = 3$ $A_{pass} = -3$ dB $A_{stop} = -100$ dB
Approximation	order (n)		
Elliptic	5	4	6
Chebyshev	7	5	7
Inverse Chebyshev	7	5	7
Butterworth	12	6	11

4. Conclusion

The governing equations of five approximation functions have been reviewed, studied and compared. Recommendation for computer aided numerical class project as an active learning strategy in the teaching of filter design has been presented and tested. Pole-zero plots and frequency response graphs have been generated as teaching aids. These plots and graphs provide more insight into the subject matter and will be very helpful for active learning. Bessel approximation is not recommendable for most applications as it suffers poor amplitude discrimination in the stopband except when linear-phase response over the passband is required. Butterworth approximation requires higher order number for any given set of filter specifications. Chebyshev and Inverse Chebyshev approximations are good but Elliptic approximation is the best as it requires the least order number although it is more complex than the others.

Compliance with ethical standards

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