Global Journal of Engineering and Technology Advances

eISSN: 2582-5003

Cross Ref DOI: 10.30574/gjeta

Journal homepage: https://gjeta.com/

(RESEARCH ARTICLE)

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S-coindex of some graph operations and its properties and conjugated polymers

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Global Journal of Engineering and Technology Advances, 2023, 16(02), 140–149

Publication history: Received on 21 June 2021; revised on 31 July 2023; accepted on 03 August 2023

Article DOI: https://doi.org/10.30574/gjeta.2023.16.2.0151

Abstract

The theory of chemical graphs is a field of mathematical chemistry that applies graph theory to the mathematical modeling of chemical phenomena. Topological index is a numerical descriptor of a molecule; it is found that there is strong correlation between the properties of chemical compounds and their molecular structure based on a specific topological feature of the corresponding molecular graph. The S-index of a graph is defined as the sum of five of the degree of vertices of a graph. In this paper, we introduce a new invariant of a graph which is identified as S-coindex. We study some basic mathematical properties and different types of graph operations like as Join, Cartesian Product, Composition, Tensor Product, Strong Product, Disjunction, Symmetric Difference, Corona Product of graphs.

Keywords: S-index; S-coindex; Graph Operations; Zagreb; F-index

1. Introduction

Topological and graph invariants based on distances between graph vertices are widely used for characterizing molecular graphs, establishing relationships between structural and property, properties of molecules, predicting biological activities of chemical compounds, and developing chemical applications. A topological index can be thought of as the conversion of a chemical structure into a real number. There are several types of topological indices, including distance-based topological indices, degree-based topological indices, and counting-related polynomials and graph indices. Graph theory has given chemists many useful tools, such as topological indices. Molecules and molecular compounds are frequently represented by molecular graphs. A molecular graph is a graph-theoretic representation of a chemical compound's structural formula, with vertices representing atoms and edges representing chemical bonds.

Topological indices have the significance of being able to be used directly as simple numerical descriptors in comparison with physical, chemical, or biological parameters of molecules in Quantitative Structure Property Relationships (QSPR) and Quantitative Structure Activity Relationships (QSAR) (QSAR). In medicinal chemistry and bioinformatics, the current trend of numerical coding of chemical structures with topological indices or topological coindices has been quite successful [1,6,9].

Let $\mathcal{G}$ and $\mathcal{H}$ be a simple connected graph with vertex sets $(V(\mathcal{G}), V(\mathcal{H}))$ and edge sets $(E(\mathcal{G}), E(\mathcal{H}))$ respectively. The degree of a vertex v, is defined as the number of vertices adjacent to v in $\mathcal{G}$ and is denoted by $\lambda_{\mathcal{G}}(v)$. The complement of a graph $\mathcal{G}$ is denoted by $\bar{\mathcal{G}}$ and is defined as the simple graph with the same vertex set $V(\mathcal{G})$ and any two vertices $uv \in E(\bar{\mathcal{G}})$ iff $uv \notin E(\mathcal{G})$. Thus $E(\mathcal{G}) \cup E(\bar{\mathcal{G}}) = E(k_n)$ and $\bar{g} = |E(\bar{\mathcal{G}})| = \frac{f(f-1)}{2} - g$. The degree of a vertex v in $\bar{\mathcal{G}}$ is given by $\lambda_{\bar{\mathcal{G}}}(v) = f - 1 - \lambda_{\mathcal{G}}(v)$ [2].

Li and Zheng [13] introduced the first general Zagreb index in 2005, defined as:

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$$M_1^{\alpha+1}(\mathcal{G}) = \sum_{v\in V(\mathcal{G})} [\lambda_{\mathcal{G}}^{\alpha+1}(v)] = \sum_{uv\in E(\mathcal{G})} [\lambda_{\mathcal{G}}^{\alpha}(u) + \lambda_{\mathcal{G}}^{\alpha}(v)]$$

I. Gutman and N. Trinajstic [8] introduced the first and second Zagreb indices of a graph in 1972, defined as:

$$M_1(\mathcal{G}) = \sum_{v\in V(\mathcal{G})} [\lambda_{\mathcal{G}}(v)^2] = \sum_{uv\in E(\mathcal{G})} [\lambda_{\mathcal{G}}(u) + \lambda_{\mathcal{G}}(v)]$$

$$M_2(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)\lambda_{\mathcal{G}}(v)]$$

Doslic T [4] introduced the Zagreb coindices of a graph in 2008, defined as

$$\bar{M}_1(\mathcal{G}) = \sum_{uv\notin E(\mathcal{G})} [\lambda_{\mathcal{G}}(u) + \lambda_{\mathcal{G}}(v)]$$

$$\bar{M}_2(\mathcal{G}) = \sum_{uv\notin E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)\lambda_{\mathcal{G}}(v)]$$

Furtula and Gutman [3] introduced the forgotten topological index (F-index) in 2015, defined as:

$$F(\mathcal{G}) = \sum_{v\in V(\mathcal{G})} [\lambda_{\mathcal{G}}(v)^3] = \sum_{uv\in E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)^2 + \lambda_{\mathcal{G}}(v)^2]$$

N. De [5,7] introduced the F-coindex in 2016, defined as:

$$\bar{F}(\mathcal{G}) = \sum_{v\notin V(\mathcal{G})} [\lambda_{\mathcal{G}}(v)^3] = \sum_{uv\notin E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)^2 + \lambda_{\mathcal{G}}(v)^2]$$

Abdu Alameri and Noman AI-Naggar [10] introduced the Y-index in 2020, defined as:

$$Y(\mathcal{G}) = \sum_{v\in V(\mathcal{G})} [\lambda_{\mathcal{G}}(v)^4] = \sum_{uv\in E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)^3 + \lambda_{\mathcal{G}}(v)^3]$$

Abdu Alameri [12] introduced the Y-coindex in 2020, defined as:

$$\bar{Y}(\mathcal{G}) = \sum_{v\notin V(\mathcal{G})} [\lambda_{\mathcal{G}}(v)^4] = \sum_{uv\notin E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)^3 + \lambda_{\mathcal{G}}(v)^3]$$

Nagarajan and Kayalvizhi [11] introduced the S-index in 2021, defined as:

$$S(\mathcal{G}) = \sum_{v\in V(\mathcal{G})} [\lambda_{\mathcal{G}}(v)^5] = \sum_{uv\in E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)^4 + \lambda_{\mathcal{G}}(v)^4]$$

In this paper, we represent some explicit expressions for the S-coindex under several graph operations and explicated some basic mathematical properties.

2. Definition: 1.1

The S-coindex of a graph $\mathcal{G}$ defined as:

$$\bar{S}(\mathcal{G}) = \sum_{v\notin V(\mathcal{G})} \lambda_{\mathcal{G}}(v)^5 = \sum_{uv\notin E(\mathcal{G})} [\lambda_{\mathcal{G}}(u)^4 + \lambda_{\mathcal{G}}(v)^4]$$

2.1. Basic Properties of the S-coindex and Conjugated Polymers

In this section, we compute the relations between the S-index and the S-coindex and derive some properties of the S-coindex of some special graphs like as complete graph, path, cycle, star graph and wheel graph.

2.1.1. Lemma: 2.1

Let $\mathcal{G}$ be a simple graph on f vertices and g edges, then

$$M_1(\bar{\mathcal{G}}) = f(f-1)^2 - 4g(f-1) + M_1(\mathcal{G}).$$

$$F(\bar{\mathcal{G}}) = f(f-1)^3 - 6g(f-1)^2 - 3(f-1)M_1(\mathcal{G}) - F(\mathcal{G}).$$

$$Y(\bar{\mathcal{G}}) = f(f-1)^4 - 8g(f-1)^3 + 6(f-1)^2 M_1(\mathcal{G}) - 4(f-1)F(\mathcal{G}) + Y(\mathcal{G}).$$

2.1.2. Proposition: 2.2

Let $\mathcal{G}$ be a simple graph on f vertices and g edges, then

$$S(\bar{\mathcal{G}}) = f(f-1)^5 - 10g(f-1)^4 + 10(f-1)^3 M_1(\mathcal{G}) - 10(f-1)^2 F(\mathcal{G}) + 5(f-1)Y(\mathcal{G}) - S(\mathcal{G}).$$

Proof:

From the definition of S-coindex, we have

$$S(\bar{\mathcal{G}}) = \sum_{v\in V(\bar{\mathcal{G}})} \lambda_{\bar{\mathcal{G}}}(v)^5 = \sum_{v\notin V(\mathcal{G})} \lambda_{\bar{\mathcal{G}}}(v)^5$$

$$= \sum_{v\in V(\mathcal{G})} \left[(f-1) - \lambda_{\mathcal{G}}(v)\right]^5$$

$$S(\bar{\mathcal{G}}) = f(f-1)^5 - 10g(f-1)^4 + 10(f-1)^3 M_1(\mathcal{G}) - 10(f-1)^2 F(\mathcal{G}) + 5(f-1)Y(\mathcal{G}) - S(\mathcal{G}).$$

We get the required result.

2.1.3. Proposition: 2.3

Let $\mathcal{G}$ be a simple graph on f vertices and g edges, then

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G}).$$

Proof:

From the definition of S-coindex, we have

$$\bar{S}(\mathcal{G}) = \sum_{uv\notin E(\mathcal{G})} \left[\lambda_{\mathcal{G}}(u)^4 + \lambda_{\mathcal{G}}(v)^4\right] = \sum_{uv\in E(\bar{\mathcal{G}})} \left[\lambda_{\mathcal{G}}(u)^4 + \lambda_{\mathcal{G}}(v)^4\right]$$

$$= \sum_{uv\in E(\bar{\mathcal{G}})} \left\{\left[(f-1) - \lambda_{\bar{\mathcal{G}}}(u)\right]^4 + \left[(f-1) - \lambda_{\bar{\mathcal{G}}}(v)\right]^4\right\}$$

$$= \sum_{uv\in E(\bar{\mathcal{G}})} \left[(f-1)^4 - 4(f-1)^3\lambda_{\bar{\mathcal{G}}}(u) + 6(f-1)^2\lambda_{\bar{\mathcal{G}}}(u)^2 - 4(f-1)\lambda_{\bar{\mathcal{G}}}(u)^3 + \lambda_{\bar{\mathcal{G}}}(u)^4\right] + \sum_{uv\in E(\bar{\mathcal{G}})} \left[(f-1)^4 - 4(f-1)^3\lambda_{\bar{\mathcal{G}}}(v) + 6(f-1)^2\lambda_{\bar{\mathcal{G}}}(v)^2 - 4(f-1)\lambda_{\bar{\mathcal{G}}}(v)^3 + \lambda_{\bar{\mathcal{G}}}(v)^4\right]$$

$$= \sum_{uv\in E(\bar{\mathcal{G}})} \left\{2(f-1)^4 - 4(f-1)^3\left[\lambda_{\bar{\mathcal{G}}}(u) + \lambda_{\bar{\mathcal{G}}}(v)\right] + 6(f-1)^2\left[\lambda_{\bar{\mathcal{G}}}(u)^2 + \lambda_{\bar{\mathcal{G}}}(v)^2\right] - 4(f-1)\left[\lambda_{\bar{\mathcal{G}}}(u)^3 + \lambda_{\bar{\mathcal{G}}}(v)^3\right] + \left[\lambda_{\bar{\mathcal{G}}}(u)^4 + \lambda_{\bar{\mathcal{G}}}(v)^4\right]\right\}$$

$$= 2\,\bar{g}(f-1)^4 - 4(f-1)^3 M_1(\bar{\mathcal{G}}) + 6(f-1)^2 F(\bar{\mathcal{G}}) - 4(f-1)Y(\bar{\mathcal{G}}) + S(\bar{\mathcal{G}})$$

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G}).$$

We get the desired result.

2.1.4. Proposition: 2.4

Let $\mathcal{G}$ be a simple graph on f vertices and g edges, then

$$\bar{S}(\bar{\mathcal{G}}) = 2g(f-1)^4 - 4(f-1)^3 M_1(\mathcal{G}) + 6(f-1)^2 F(\mathcal{G}) - 4(f-1)Y(\mathcal{G})S(\mathcal{G}).$$

Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\bar{\mathcal{G}}) = (f-1)Y(\bar{\mathcal{G}}) - S(\bar{\mathcal{G}})$$

By using the Lemma 2.1(3) and Proposition 2.2, we get the desired result.

2.1.5. Corollary: 2.5

In this part, the S-coindex of some special graphs are calculated.

- For a complete graph K_n, with n vertices:

$$\bar{S}(K_n) = 0, n \geq 3$$

- For a cycle graph C_n, with n vertices:

$$\bar{S}(C_n) = 16n(n-3)\,, n \geq 3$$

- For a path graph P_n, with n vertices:

$$\bar{S}(P_n) = 16n^2 - 78n + 92\,, n \geq 3$$

- For a star graph S_n, with n vertices:

$$\bar{S}(S_n) = n^2 - 3n + 2\,, n \geq 3$$

- For a wheel graph W_n, with n vertices:

$$\bar{S}(W_n) = 81n^2 - 324n - n^4\,, n \geq 3$$

3. Conjugated Polymers of BODIPY

Let $\mathcal{G}$ be a molecular graph with vertices representing atoms and edges representing bonds joining the atoms. The numerical value d_v represents the vertex degree at which the sum of the edges at v. $|V_u|$ denotes the cardinality of the molecular graph's vertices, while $|E_{u,v}|$ denotes the cardinality of the edges connected by the vertices u and v.

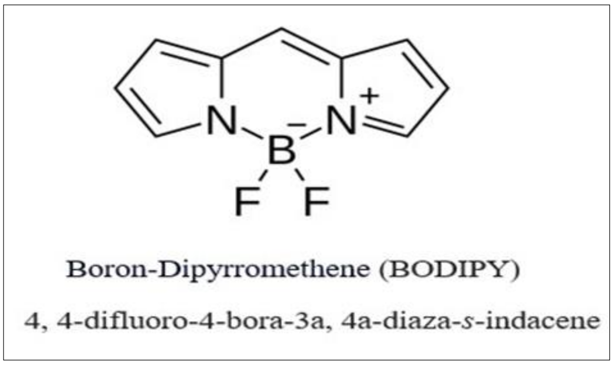

Figure 1 Molecular Structure of BODIPY (Monomer)

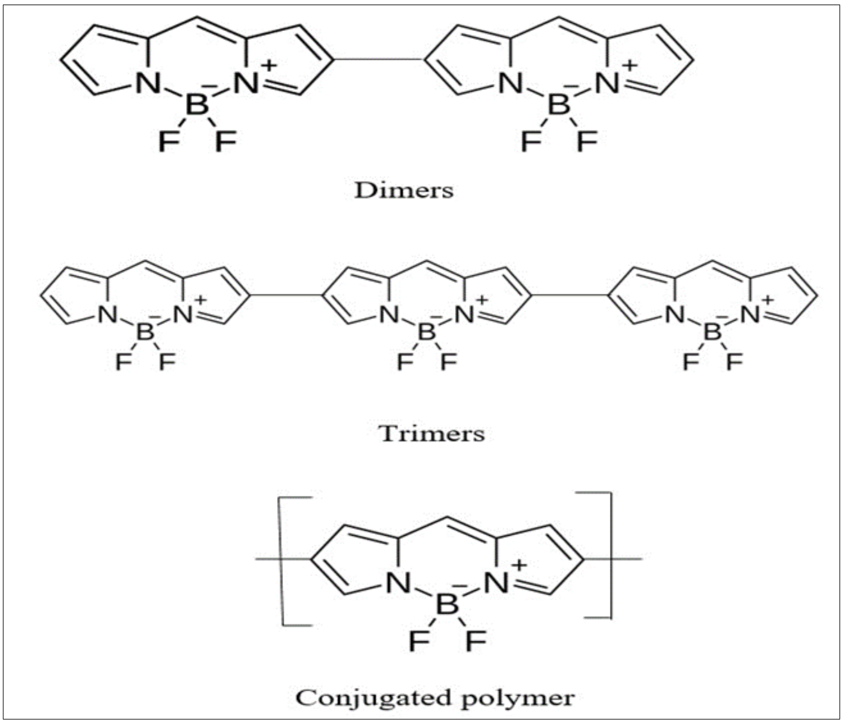

Figure 2 Molecular Structure of BODIPY [n-polymer]

Table 1 and 2 show the vertex and edge partitions based on degree of BODIPY conjugated polymer.

Table 1 Vertex partition based on degree

$\lvert V_u \rvert$	Cardinality
$\lvert V_1 \rvert$	2n
$\lvert V_2 \rvert$	5n+2
$\lvert V_3 \rvert$	6n-2
$\lvert V_4 \rvert$	N
Number of vertices	14n

Table 2 Edge partition based on degree

$\lvert E_{u,v} \rvert$	Cardinality
$\lvert E_{2,2} \rvert$	4
$\lvert E_{2,3} \rvert$	10n-4
$\lvert E_{3,3} \rvert$	3n-1
$\lvert E_{3,4} \rvert \cup \lvert E_{1,4} \rvert$	4n
Number of edges	17n-1

3.1. Theorem 2.6

If $\mathcal{G}$ be a graph with n vertices and edges, then

- $M_1(\mathcal{G}) = 92n - 10$
- $M_2(\mathcal{G}) = 119n - 17$
- $F(\mathcal{G}) = 268n - 38$
- $Y(\mathcal{G}) = 824n - 130$
- $S(\mathcal{G}) = 2644n - 422$

3.2. Theorem 2.7

If $\mathcal{G}$ be a graph with n vertices and edges, then

- $\overline{M_1}(\mathcal{G}) = 322n + 12$
- $\overline{M_2}(\mathcal{G}) = 578n^2 - 233n + 24$
- $\bar{F}(\mathcal{G}) = 788n + 48$
- $\bar{Y}(\mathcal{G}) = 2128n + 168$
- $\bar{S}(\mathcal{G}) = 6248n + 552$

4. Main Results

In this section, we will study about the S-coindex of different types of graph operations.

4.1. Join

The join $\mathcal{G} + \mathcal{H}$ of graphs $\mathcal{G}$ and $\mathcal{H}$ with vertex sets $V(\mathcal{G})$ and $V(\mathcal{H})$ and edge sets $E(\mathcal{G})$ and $E(\mathcal{H})$ is the graph union $\mathcal{G} \cup \mathcal{H}$ together with all the edges between $V(\mathcal{G})$ and $V(\mathcal{H})$. Obviously, $|V(\mathcal{G} + \mathcal{H})| = |V(\mathcal{G})| + |V(\mathcal{H})| = f_1 + f_2$and $|E(\mathcal{G} + \mathcal{H})| = |E(\mathcal{G})| + |E(\mathcal{H})| + |V(\mathcal{G})||V(\mathcal{H})| = g_1 + g_2 + f_1 f_2$.

$$\lambda_{\mathcal{G}+\mathcal{H}}(v) = \begin{Bmatrix} \lambda_{\mathcal{G}}(v) + f_2, v \in V(\mathcal{G}) \\ \lambda_{\mathcal{H}}(v) + f_1, v \in V(\mathcal{H}) \end{Bmatrix}$$

4.2. Theorem: 3.1

The S-coindex of $\mathcal{G} + \mathcal{H}$ is given by

$$\bar{S}(\mathcal{G} + \mathcal{H}) = \bar{S}(\mathcal{G}) + \bar{S}(\mathcal{H}) + 4[f_2\bar{Y}(\mathcal{G}) + f_1\bar{Y}(\mathcal{H}) + f_2^3\overline{M_1}(\mathcal{G}) + f_1^3\overline{M_1}(\mathcal{H})] + 6f_2^2\bar{F}(\mathcal{G}) + 6f_1^2\bar{F}(\mathcal{H}) + 2[f_2^4\overline{g_1} + f_1^4\overline{g_2}].$$

4.2.1. Proof

From the definition of S-Coindex, we have

$$\bar{S}(\mathcal{G} + \mathcal{H}) = \sum_{uv \notin E(\mathcal{G}+\mathcal{H})} \left[\lambda_{\mathcal{G}+\mathcal{H}}(u)^4 + \lambda_{\mathcal{G}+\mathcal{H}}(v)^4\right]$$

$$= \sum_{uv \notin E(\mathcal{G})} \left[\lambda_{\mathcal{G}+\mathcal{H}}(u)^4 + \lambda_{\mathcal{G}+\mathcal{H}}(v)^4\right] + \sum_{uv \notin E(\mathcal{H})} \left[\lambda_{\mathcal{G}+\mathcal{H}}(u)^4 + \lambda_{\mathcal{G}+\mathcal{H}}(v)^4\right]$$

$$= A + B$$

From 'A' we have

$$\sum_{uv \notin E(\mathcal{G})} \left[\lambda_{\mathcal{G}+\mathcal{H}}(u)^4 + \lambda_{\mathcal{G}+\mathcal{H}}(v)^4\right] = \sum_{uv \notin E(\mathcal{G})} \left[\left(\lambda_{\mathcal{G}}(u) + f_2\right)^4 + \left(\lambda_{\mathcal{G}}(v) + f_2\right)^4\right]$$

$$A = \bar{S}(\mathcal{G}) + 4f_2\bar{Y}(\mathcal{G}) + 6f_2^2\bar{F}(\mathcal{G}) + 4f_2^3\overline{M_1}(\mathcal{G}) + 2f_2^4\overline{g_1}.$$

From 'B' we have

Similarly, we get

$$B = \bar{S}(\mathcal{H}) + 4f_1\bar{Y}(\mathcal{H}) + 6f_1^2\bar{F}(\mathcal{H}) + 4f_1^3\overline{M_1}(\mathcal{H}) + 2f_1^4\overline{g_2}.$$

Adding A and B, we get the desired result.

4.2.2. Cartesian product

The Cartesian product $\mathcal{G} \times \mathcal{H}$ of graphs $\mathcal{G}$ and $\mathcal{H}$ has the vertex set $V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$ and $(u,x)(v,y)$ is an edge of $\mathcal{G} \times \mathcal{H}$ if $uv \in E(\mathcal{G})$ and $x = y$, or $u = v$ and $xy \in E(\mathcal{H})$.Obviously, $|V(\mathcal{G} \times \mathcal{H})| = |V(\mathcal{G})||V(\mathcal{H})| = f_1f_2$ and $|E(\mathcal{G} \times \mathcal{H})| = |E(\mathcal{G})||V(\mathcal{H})| + |E(\mathcal{H})||V(\mathcal{G})| = g_1f_2 + g_2f_1$.

$$\lambda_{\mathcal{G}\times\mathcal{H}}(a,b) = \lambda_{\mathcal{G}}(a) + \lambda_{\mathcal{G}}(b)$$

4.3. Theorem: 3.2

The S-coindex of $\mathcal{G} \times \mathcal{H}$ is given by

$$\bar{S}(\mathcal{G} \times \mathcal{H}) = (f_1f_2 - 1)[f_2Y(\mathcal{G}) + f_1Y(\mathcal{H}) + 8g_1F(\mathcal{G}) + 8g_2F(\mathcal{H}) + 6M_1(\mathcal{G})M_1(\mathcal{H})] - f_2S(\mathcal{G}) - f_1S(\mathcal{H}) - 10Y(\mathcal{G})g_2 - 10F(\mathcal{G})M_1(\mathcal{H}) - 10F(\mathcal{H})M_1(\mathcal{G}) - 10g_1Y(\mathcal{H}) .$$

4.3.1. Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\mathcal{G} \times \mathcal{H}) = (f_1f_2 - 1)Y(\mathcal{G} \times \mathcal{H}) - S(\mathcal{G} \times \mathcal{H})$$

By using the expression $Y(\mathcal{G} \times \mathcal{H})$ explained in [10] and $S(\mathcal{G} \times \mathcal{H})$ explained in [11], we get the desired result.

4.3.2. Composition

The Composition $\mathcal{G}[\mathcal{H}]$ of graphs $\mathcal{G}$ and $\mathcal{H}$ with disjoint vertex sets $V(\mathcal{G})$ and $V(\mathcal{H})$ and edge sets $E(\mathcal{G})$ and $E(\mathcal{H})$ is the graph with vertex set $V(\mathcal{G}) \times V(\mathcal{H})$ and $u = (u_1, v_1)$ is adjacent to $v = (u_2, v_2)$ whenever u_1 is adjacent to u_2 or $u_1 = u_2$ and v_1 is adjacent to v_2 . $|V(\mathcal{G}[\mathcal{H}])| = |V(\mathcal{G})||V(\mathcal{H})| = f_1f_2$, $|E(\mathcal{G}[\mathcal{H}])| = |E(\mathcal{G})||V(\mathcal{H})|^2 + |V(\mathcal{G})||E(\mathcal{H})| = g_1f_2^2 + f_1g_2$.

$$\lambda_{\mathcal{G}[\mathcal{H}]}(a,b) = f_2\lambda_{\mathcal{G}}(a) + \lambda_{\mathcal{H}}(b)$$

4.4. Theorem: 3.3

The S-coindex of $\mathcal{G}[\mathcal{H}]$ is given by

$$\bar{S}(\mathcal{G}[\mathcal{H}]) = (f_1f_2 - 1)[f_2^5Y(\mathcal{G}) + f_1Y(\mathcal{H}) + 8g_1f_2F(\mathcal{H}) + 8g_2f_2^3F(\mathcal{G}) + 6f_2^2M_1(\mathcal{G})M_1(\mathcal{H})] - f_2^6S(\mathcal{G}) - f_1S(\mathcal{H}) - 10f_2^3F(\mathcal{G})M_1(\mathcal{H}) - 10f_2^2F(\mathcal{H})M_1(\mathcal{G}) - 10g_2f_2^4Y(\mathcal{G}).$$

4.4.1. Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\mathcal{G}[\mathcal{H}]) = (f_1f_2 - 1)Y(\mathcal{G}[\mathcal{H}]) - S(\mathcal{G}[\mathcal{H}])$$

By using the expression $Y(\mathcal{G}[\mathcal{H}])$ explained in [10] and $S(\mathcal{G}[\mathcal{H}])$ explained in [11], we get the desired result.

4.4.2. Tensor product

The Tensor product $\mathcal{G}\otimes\mathcal{H}$ of graphs $\mathcal{G}$ and $\mathcal{H}$ has the vertex set $V(\mathcal{G}\otimes\mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$ and $(u,x)(v,y)$ is an edge of $\mathcal{G} \oplus \mathcal{H}$ if $uv \in E(\mathcal{G})$ and $xy \in E(\mathcal{H})$. Obviously, $|V(\mathcal{G}\otimes\mathcal{H})| = |V(\mathcal{G})||V(\mathcal{H})| = f_1f_2$, $|E(\mathcal{G}\otimes\mathcal{H})| = 2|E(\mathcal{G})||E(\mathcal{H})| = 2g_1g_2$ and

$$\lambda_{\mathcal{G}\otimes\mathcal{H}}(u,x) = \lambda_{\mathcal{G}}(u)\lambda_{\mathcal{H}}(x).$$

4.5. Theorem: 3.4

The S-coindex of $\mathcal{G}\otimes\mathcal{H}$ is given by

$\bar{S}(\mathcal{G}\otimes\mathcal{H}) = f_1 f_2 Y(\mathcal{G})Y(\mathcal{H}) - Y(\mathcal{G})Y(\mathcal{H}) - S(\mathcal{G})S(\mathcal{H})$.

4.5.1. Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\mathcal{G}\otimes\mathcal{H}) = (f_1 f_2 - 1)Y(\mathcal{G}\otimes\mathcal{H}) - S(\mathcal{G}\otimes\mathcal{H})$$

By using the expression $Y(\mathcal{G}\otimes\mathcal{H})$ explained in [10] and $S(\mathcal{G}\otimes\mathcal{H})$ explained in [11], we get the desired result.

4.5.2. Strong product

The Strong product $\mathcal{G} * \mathcal{H}$ of a graphs $\mathcal{G}$ and $\mathcal{H}$ is a graph with vertex set $V(\mathcal{G}) \times V(\mathcal{H})$ and any two vertices (u_p, v_r) and (u_q, v_s) are adjacent if and only if $[u_p = u_q \text{ and } v_r v_s \in E(\mathcal{H})]$ or $[v_r = v_s \text{ and } u_p u_q \in E(\mathcal{G})]$ or $[u_p u_q \in E(\mathcal{G}) \text{ and } v_r v_s \in E(\mathcal{H})]$. $|V(\mathcal{G} * \mathcal{H})| = |V(\mathcal{G})||V(\mathcal{H})| = f_1 f_2$, $|E(\mathcal{G} * \mathcal{H})| = |E(\mathcal{G})||V(\mathcal{H})| + |V(\mathcal{G})||E(\mathcal{H})| + 2|E(\mathcal{G})||E(\mathcal{H})| = g_1 f_2 + f_1 g_2 + 2g_1 g_2$.

$$\lambda_{\mathcal{G}*\mathcal{H}}((a,b)) = \lambda_{\mathcal{G}}(a) + \lambda_{\mathcal{H}}(b) + \lambda_{\mathcal{G}}(a)\lambda_{\mathcal{H}}(b)$$

4.6. Theorem: 3.5

The S-coindex of $\mathcal{G} * \mathcal{H}$ is given by

$$\bar{S}(\mathcal{G} * \mathcal{H}) = (f_1 f_2 - 1)\{Y(\mathcal{G})[4F(\mathcal{H}) + 6M_1(\mathcal{H}) + 8g_2 + f_2] + 4F(\mathcal{G})[3M_1(\mathcal{H}) + 2g_2] + Y(\mathcal{H})[4F(\mathcal{G}) + 6M_1(\mathcal{G}) + 8g_1 + f_1] + 4F(\mathcal{H})[3M_1(\mathcal{G}) + 2g_1] + Y(\mathcal{G})Y(\mathcal{H}) + 12F(\mathcal{G})F(\mathcal{H}) + 6M_1(\mathcal{G})M_1(\mathcal{H})\} - S(\mathcal{G})f_2 - S(\mathcal{H})f_1 - S(\mathcal{G})S(\mathcal{H}) - 10Y(\mathcal{G})g_2 - 10S(\mathcal{G})g_2 - 10Y(\mathcal{H})g_1 - 10S(\mathcal{H})g_1 - 5S(\mathcal{G})Y(\mathcal{H}) - 5Y(\mathcal{G})S(\mathcal{H}) - 10F(\mathcal{G})M_1(\mathcal{H}) - 20Y(\mathcal{G})M_1(\mathcal{H}) - 10S(\mathcal{G})M_1(\mathcal{H}) - 10F(\mathcal{H})M_1(\mathcal{G}) - 20Y(\mathcal{H})M_1(\mathcal{G}) - 10S(\mathcal{H})M_1(\mathcal{G}) - 10S(\mathcal{G})F(\mathcal{H}) - 20Y(\mathcal{G})Y(\mathcal{H}) - 10F(\mathcal{G})S(\mathcal{H}) - 30F(\mathcal{G})F(\mathcal{H}) - 30Y(\mathcal{G})M_1(\mathcal{H}) - 30F(\mathcal{G})Y(\mathcal{H}).$$

4.6.1. Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\mathcal{G} * \mathcal{H}) = (f_1 f_2 - 1)Y(\mathcal{G} * \mathcal{H}) - S(\mathcal{G} * \mathcal{H})$$

By using the expression $Y(\mathcal{G} * \mathcal{H})$ explained in [10] and $S(\mathcal{G} * \mathcal{H})$ explained in [11], we get the desired result.

4.6.2. Disjunction

The Disjunction $\mathcal{G} \vee \mathcal{H}$ of a graphs $\mathcal{G}$ and $\mathcal{H}$ is the graph with vertex set $V(\mathcal{G}) \times V(\mathcal{H})$ and $u_1 v_1$ is adjacent with $u_2 v_2$ whenever $u_1 u_2 \in E(\mathcal{G})$ and $v_1 v_2 \in E(\mathcal{H})$ $|V(\mathcal{G} \vee \mathcal{H})| = |V(\mathcal{G})||V(\mathcal{H})| = f_1 f_2$, $|E(\mathcal{G} \vee \mathcal{H})| = |E(\mathcal{G})||V(\mathcal{H})|^2 + |E(\mathcal{H})||V(\mathcal{G})|^2 - 2|E(\mathcal{G})||E(\mathcal{H})| = g_1 f_2^2 + g_2 f_1^2 - 2g_1 g_2$.

$$\lambda_{\mathcal{G}\vee\mathcal{H}}((a,b)) = f_2\lambda_{\mathcal{G}}(a) + f_1\lambda_{\mathcal{H}}(b) - \lambda_{\mathcal{G}}(a)\lambda_{\mathcal{H}}(b)$$

4.7. Theorem: 3.6

The S-coindex of $\mathcal{G} \vee \mathcal{H}$ is given by

$$\bar{S}(\mathcal{G} \vee \mathcal{H}) = (f_1 f_2 - 1)\{f_1 Y(\mathcal{H})[f_1^4 - 4F(\mathcal{G}) + 6f_1 M_1(\mathcal{G}) - 8g_1 f_1^2] + 4f_1^2 f_2 F(\mathcal{H})[2g_1 f_1 - 3M_1(\mathcal{G})] + f_2 Y(\mathcal{G})[f_2^4 - 4F(\mathcal{H}) + 6f_2 M_1(\mathcal{H}) - 8g_2 f_2^2] + 4f_2^2 f_1 F(\mathcal{G})[2g_2 f_2 - 3M_1(\mathcal{H})] + Y(\mathcal{G})Y(\mathcal{H}) + 12f_1 f_2 F(\mathcal{G})F(\mathcal{H}) +$$

$$6f_1^2f_2^2M_1(\mathcal{G})M_1(\mathcal{H})\} - f_2^5S(\mathcal{G}) - f_1^5S(\mathcal{H}) + S(\mathcal{G})S(\mathcal{H}) - 10f_2^4f_1Y(\mathcal{G})g_2 + 10f_2^4S(\mathcal{G})g_2 - 10f_2^3F(\mathcal{G})f_1^2M_1(\mathcal{H}) - 10f_2^3S(\mathcal{G})M_1(\mathcal{H}) - 10f_2^2F(\mathcal{H})f_1^3M_1(\mathcal{G}) - 10f_2g_1f_1^4Y(\mathcal{H}) + 10g_1f_1^4S(\mathcal{H}) - 10f_1^3S(\mathcal{H})M_1(\mathcal{G}) + 10f_2^2S(\mathcal{G})F(\mathcal{H}) + 10f_1^2S(\mathcal{H})F(\mathcal{G}) + 20f_2^3Y(\mathcal{G})f_1M_1(\mathcal{H}) + 20f_1^3Y(\mathcal{H})f_2M_1(\mathcal{G}) + 20f_2f_1Y(\mathcal{G})Y(\mathcal{H}) - 5f_2S(\mathcal{G})Y(\mathcal{H}) - 5f_1Y(\mathcal{G})S(\mathcal{H}) - 30f_2^2f_1Y(\mathcal{G})F(\mathcal{H}) - 30f_1^2f_2F(\mathcal{G})Y(\mathcal{H}) + 30f_1^2f_2^2F(\mathcal{G})F(\mathcal{H}).$$

4.7.1. Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\mathcal{G} \vee \mathcal{H}) = (f_1f_2 - 1)Y(\mathcal{G} \vee \mathcal{H}) - S(\mathcal{G} \vee \mathcal{H})$$

By using the expression $Y(\mathcal{G} \vee \mathcal{H})$ explained in [10] and $S(\mathcal{G} \vee \mathcal{H})$ explained in [11], we get the desired result.

4.7.2. Symmetric Difference

The Symmetric Difference $\mathcal{G} \oplus \mathcal{H}$ of two graphs $\mathcal{G}$ and $\mathcal{H}$ is a graph with vertex set $V(\mathcal{G}) \times V(\mathcal{H})$ and $E(\mathcal{G} \oplus \mathcal{H}) = \{(u_1, u_2)(v_1, v_2)/u_1v_1 \in E(\mathcal{G}) \text{ or } u_2v_2 \in E(\mathcal{H}) \text{ but not both}\}$

$$|V(\mathcal{G} \oplus \mathcal{H})| = |V(\mathcal{G})||V(\mathcal{H})| = f_1f_2,\ |E(\mathcal{G} \oplus \mathcal{H})| = |E(\mathcal{G})||V(\mathcal{H})|^2 + |E(\mathcal{H})||V(\mathcal{G})|^2 - 4|E(\mathcal{G})||E(\mathcal{H})| = g_1f_2^2 + g_2f_1^2 - 4g_1g_2.$$

$$\lambda_{\mathcal{G} \oplus \mathcal{H}}((a,b)) = f_2\lambda_{\mathcal{G}}(a) + f_1\lambda_{\mathcal{H}}(b) - 2\lambda_{\mathcal{G}}(a)\lambda_{\mathcal{H}}(b)$$

4.8. Theorem: 3.7

The S-coindex of $\mathcal{G} \oplus \mathcal{H}$ is given by

$$\bar{S}(\mathcal{G} \oplus \mathcal{H}) = (f_1f_2 - 1)\{f_1Y(\mathcal{H})[f_1^4 - 32F(\mathcal{G}) + 24f_1M_1(\mathcal{G}) - 16g_1f_1^2] + 8f_1^2f_2F(\mathcal{H})[g_1f_1 - 3M_1(\mathcal{G})] + f_2Y(\mathcal{G})[f_2^4 - 32F(\mathcal{H}) + 24f_2M_1(\mathcal{H}) - 16g_2f_2^2] + 8f_2^2f_1F(\mathcal{G})[g_2f_2 - 3M_1(\mathcal{H})] + 16Y(\mathcal{G})Y(\mathcal{H}) + 48f_1f_2F(\mathcal{G})F(\mathcal{H}) + 6f_1^2f_2^2M_1(\mathcal{G})M_1(\mathcal{H})\} - f_2^5S(\mathcal{G}) - f_1^5S(\mathcal{H}) + 32S(\mathcal{G})S(\mathcal{H}) - 10f_2^4f_1Y(\mathcal{G})g_2 + 20f_2^4S(\mathcal{G})g_2 - 10f_2^3F(\mathcal{G})f_1^2M_1(\mathcal{H}) - 40f_2^3S(\mathcal{G})M_1(\mathcal{H}) - 10f_2^2F(\mathcal{H})f_1^3M_1(\mathcal{G}) - 10f_2g_1f_1^4Y(\mathcal{H}) + 20g_1f_1^4S(\mathcal{H}) - 40f_1^3S(\mathcal{H})M_1(\mathcal{G}) + 80f_2^2S(\mathcal{G})F(\mathcal{H}) + 80f_1^2S(\mathcal{H})F(\mathcal{G}) + 40f_2^3Y(\mathcal{G})f_1M_1(\mathcal{H}) + 40f_1^3Y(\mathcal{H})f_2M_1(\mathcal{G}) + 160f_2f_1Y(\mathcal{G})Y(\mathcal{H}) - 80f_2S(\mathcal{G})Y(\mathcal{H}) - 80f_1Y(\mathcal{G})S(\mathcal{H}) - 120f_2^2f_1Y(\mathcal{G})F(\mathcal{H}) - 120f_1^2f_2F(\mathcal{G})Y(\mathcal{H}) + 120f_1^2f_2^2F(\mathcal{G})F(\mathcal{H}).$$

4.8.1. Proo

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f-1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}\left(\mathcal{G} \bigoplus \mathcal{H}\right) = (f_1f_2 - 1)Y\left(\mathcal{G} \bigoplus \mathcal{H}\right) - S\left(\mathcal{G} \bigoplus \mathcal{H}\right)$$

By using the expression $Y(\mathcal{G} \oplus \mathcal{H})$ explained [10] and $S(\mathcal{G} \oplus \mathcal{H})$ explained in [11], we get the desired result.

4.8.2. Corona product

The Corona product $\mathcal{G} \odot \mathcal{H}$ of graphs $\mathcal{G}$ and $\mathcal{H}$ with disjoint vertex sets $V(\mathcal{G})$ and $V(\mathcal{H})$ and edge sets $E(\mathcal{G})$ and $E(\mathcal{H})$ is the graph obtained by one copy of $\mathcal{G}$ and f_1 copies of $\mathcal{H}$ and joining the i^{th} vertex of $\mathcal{G}$ to every vertex in i^{th} copy of $\mathcal{H}$. Obviously, $|V(\mathcal{G} \odot \mathcal{H})| = |V(\mathcal{G})| + |V(\mathcal{G})||V(\mathcal{H})| = f_1 + f_1f_2$, $|E(\mathcal{G} \odot \mathcal{H})| = |E(\mathcal{G})| + |V(\mathcal{G})||E(\mathcal{H})| + |V(\mathcal{G})||V(\mathcal{H})| = g_1 + f_1g_2 + f_1f_2$.

$$\lambda_{\mathcal{G} \odot \mathcal{H}}(v) = \begin{Bmatrix} \lambda_{\mathcal{G}}(v) + f_2, v \in V(\mathcal{G}) \\ \lambda_{\mathcal{H}}(v) + 1, v \in V(\mathcal{H}) \end{Bmatrix}$$

4.9. Proposition: 3.8

The Y-index of $\mathcal{G} \odot \mathcal{H}$ is given by

$Y(\mathcal{G}\odot\mathcal{H}) = Y(\mathcal{G}) + 4f_2F(\mathcal{G}) + 6f_2^2M_1(\mathcal{G}) + 8f_2^3g_1 + f_1f_2^4 + f_1[Y(\mathcal{H}) + 4F(\mathcal{H}) + 6M_1(\mathcal{H}) + 8g_2 + f_2]$.

4.10. Theorem: 3.9

The S-coindex of $\mathcal{G}\odot\mathcal{H}$ is given by

$$\bar{S}(\mathcal{G}\odot\mathcal{H}) = (f_1f_2 + f_1 - 1)\{Y(\mathcal{G}) + 4f_2F(\mathcal{G}) + 6f_2^2M_1(\mathcal{G}) + 8f_2^3g_1 + f_1f_2^4 + f_1[Y(\mathcal{H}) + 4F(\mathcal{H}) + 6M_1(\mathcal{H}) + 8g_2 + f_2]\} - S(\mathcal{G}) - f_1S(\mathcal{H}) - 5f_2Y(\mathcal{G}) - 5f_1Y(\mathcal{H}) - 10f_2^2F(\mathcal{G}) - 10f_2^3M_1(\mathcal{G}) - 10f_2^4g_1 - f_2^4f_1 - 10f_1F(\mathcal{H}) - 10f_1M_1(\mathcal{H}) + 10f_1g_2 + f_1f_2.$$

4.10.1. Proof

From the Proposition 2.3, we have

$$\bar{S}(\mathcal{G}) = (f - 1)Y(\mathcal{G}) - S(\mathcal{G})$$

$$\bar{S}(\mathcal{G}\odot\mathcal{H}) = (f_1f_2 + f_1 - 1)Y(\mathcal{G}\odot\mathcal{H}) - S(\mathcal{G}\odot\mathcal{H})$$

By using the expression $Y(\mathcal{G}\odot\mathcal{H})$ explained in proposition 3.8 and $S(\mathcal{G}\odot\mathcal{H})$ explained in [11], we get the desired result.

5. Conclusion

Topological indices are defined and used in many fields to investigate the properties of various objects such as atoms and molecules. Mathematicians and chemists have defined and studied a number of topological indices. In this study, we discussed the S-coindex of some basic mathematical properties and expressed the different graph operations such as Join, Cartesian Product, Composition, Tensor Product, Strong Product, Disjunction, Symmetric Difference, Corona Product of graphs are obtained.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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