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Solvibility of the Kahn-Hillard model in Quasi-Sobolev space

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Abstract

In this paper, a theoretical investigation of the Kahn-Hillard Model In Quasi-Sobolev Space is fulfilled by constructing a smooth space of a Kahn-Hillard Model in a bounded domain with a smooth boundary.

Keywords: Growth-diffusion problem; Modified finite difference method; Central difference; Non-classical variational

1. Introduction

The article will study an analytical solution of the Kahn-Hillard Model

$$\frac{\partial u}{\partial t} = \Delta[F'(u) - \epsilon^2 \Delta u] \quad \text{..................(1)}$$

And the boundary conditions:

$$\text{and } u = \Delta u = 0 \qquad \text{.............(2)}$$

Where $\partial\Omega \times [0, T]$, and $F(u)$ is the Helmholtz value of the free energy of a molecule of a homogeneous system with composition u. Often the energy gradient coefficient is referred to $(k = \epsilon^2, 0 < \epsilon^2 \ll 1)$, which is related to the interfacial energy, $u(x,t)$ is the concentration of one of the two components $(x,t) \in \Omega \times \mathbb{R}_+; \Omega = (0, L)$. Concentration should be understood either as a volume fraction or as a mass fraction, depending on which physical system is being studied.

A problem (1) – (2) is a mathematical physics problem that describes the process of phase separation, i.e. the mechanism by which a mixture of two or more substances is separated into separate regions with different chemical compositions and physical properties.

Let U – be a Banach space and $L(U)$ – be the space of linear bounded operators. A mapping $\mathrm{U} \in C(U; L(U))$ is called a semigroup of operators if, for all $s, t \in R_+$

$$\mathrm{U}^s \mathrm{U}^t = \mathrm{U}^{s+t}. \qquad \text{...........(3)}$$

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Usually a semigroup of operators is identified with its graph . A semigroup $\{U^t: t \in R_+\}$ is said to be holomorphic if it is analytically continuable with property (3) preserved to some sector of the complex plane containing the semiaxis R_+ A holomorphic semigroup is said to be degenerate if its identity $P=s-\lim_{t\to 0+} U^t$ is a projection to *U*.

Holomorphic degenerate semigroups of operators first appeared in [1], [2] as resolving semigroups of linear evolution equations of Sobolev type

$$L\dot{u}=Mu, \qquad \text{…………(4)}$$

where the operator $L \in L(U;F)$ (i.e., linear and bounded) and the operator $M \in Cl(U;F)$ (i.e., linear, closed, and densely defined) F is another Banach space. A complete theory of such semigroups is presented [3]; the theory is extended to Fréchet spaces [4,5].

2. Analytical investigation of linear closed operators in quasi-Banach spaces

Let U be a lineal over a field R . An ordered pair ($U, {}_U\|\cdot\|$) is called a quasi-normed space if the function ${}_U\|\cdot\|: U \to R$ satisfies the following conditions:

- ${}_U\|u\| \geq 0$ for all $u \in U$, and ${}_U\|u\|=0$ exactly when $u=0$, where 0 is the zero of the lineal U;
- ${}_U\|\alpha u\|=|\alpha|_U \|u\|$ for all $u \in U$, $\alpha \in R$;
- ${}_U\|u+v\|=C({}_U\|u\|+_U\|v\|)$ for all $u, v \in U$ where a constant $C \geq 1$.

A function ${}_U\|u\|$ with properties (1)–(3) is called a quasi-norm. It is obvious that in the case $C=1$ this function will be the norm. A quasi-Banach space is a metrizable complete quasi-normed space. A well-known example of quasi-Banach spaces are the spaces of sequences $\ell_q, \ q \in (0,1)$ (for $q \in [1,+\infty)$ the spaces are Banach). Let here and below be a monotone sequence such that

$$\ell_q^m = \left\{ u = \{u_k\}: \sum_{k=1}^{\infty} \left(\lambda_k^{\frac{m}{2}} |u_k| \right)^q < +\infty \right\}$$

With the quasi-norm ${}_q^m\|u\| = \left(\sum_{k=1}^{\infty} \left(\lambda_k^{\frac{m}{2}} |u_k| \right)^q \right)^{1/q}$, $m \in R$. Obviously, when $q \in [1,+\infty)$, the space ℓ_q^m Banach space, $\ell_q^0 = \ell_q$ and there are dense and continuous embeddings ℓ_q^n in ℓ_q^m where $n > m$ and $q \in R_+$.

Let *U* and *F* be quasi-Banach spaces, a linear operator $L: U \to F$ is called continuous if $\lim_{k\to\infty} Lu_k = L\left(\lim_{k\to\infty} u_k\right)$ for any sequence $\{u_k\} \subset U$, converging to *U*. It is easy to show that a linear operator $L: U \to F$ is continuous exactly when it is bounded (i.e. maps bounded sets to bounded ones). A Lineal $L(U;F)$ of linear bounded operators is a quasi-Banach space with a quasi-norm ${}_{L(U;F)}\|L\| = \sup_{{}_U\|u\|=1} {}^F\|Lu\|$, where ${}_U\|\cdot\|$ (${}_F\|\cdot\|$) is a quasi-norm in U (F) . A sequence $\{L_k\} \subset L(U;F)$ is called strongly convergent to the operator $L \in L(U;F)$ if for any $u \in U$, ${}_F\|L_k u - Lu\| \to 0$, $k \to \infty$; and uniformly convergent if ${}_{L(U;F)}\|L_k - L\| \to 0$, $k \to \infty$.

Theorem 1. (analog of the Banach–Steinhaus theorem).

The sequence $\{L_k\} \subset L(U;F)$ converges uniformly to an operator $L \in L(U;F)$ on some lineal U^0 dense in U exactly when

- The sequence $\{L_k\}$ is limited;
- The sequence $\{L_k\}$ strongly converges to L on U^0.

A linear operator $L:U \to F$ is said to be closed if its graph $graphL=\{(u, f) \in U \times F: f=Lu\}$ is closed in the quasi-norm ${}_{graphL}\|u\| = {}_U\|u\| + {}_F\|Lu\|$.

Theorem 2. If the operator $L \in L(U;F)$, then L is a closed operator.

Theorem 3. Let the linear operator $L:U \to F$ be closed and the domain $domL = U$ then $L \in L(U;F)$.

Theorem 4. Let the operator $L:U \to F$ be closed, and let there $L^{-1}: F \to U$ be an operator . Then L^{-1} is a closed operator.

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3. Evolutionary equations of Sobolev type

Let U and F be quasi-Banach spaces; $\{U^t:t \in R_+\}$ and $\{F^t:t \in R_+\}$ are holomorphic degenerate operator semigroups defined on the spaces U and F respectively. Then there are projectors $P=s-\lim_{t\to 0+} U^t$ and $Q=s-\lim_{t\to 0+} F^t$, which split the spaces U and F into direct sums $U = U^0 \oplus U^1$, and $(F = F^0 \oplus F^1)$, where $U^0=kerP$, $U^1=imP$, $F^0=kerQ$, $F^1=imQ$.

Let U and F be quasi-Banach spaces of sequences, and let the operators $L \in L(U;F)$ and $M \in Cl(U;F)$ be constructed. Consider the linear evolution equation of the Sobolev type

$$L\dot{u}=Mu. \qquad \text{..........(5)}$$

A vector function $u \in C^1(R_+;U)$ satisfying (5) pointwise is called a (classical) solution of this equation. A solution $u = u(t)$ of equation (5) is called a solution of the weakened Cauchy problem (according to S.G. Krein) if for some $u_0 \in U$ satisfied

$$\lim_{t\to 0+} u(t) = u_0. \qquad \text{............(6)}$$

Definition 1. A set is called the phase space of equation (5) if

- Any solution $u = u(t)$ of equation (5) lies in P pointwise, i.e. $u(t) \in P$ for any $t \in R_+$;
- For any $u_0 \in P$, there exists a unique solution to problem (5), (6).

Theorem 5. Let the operators M and L be defined as above. Then the phase space of Eq. (5) is the subspace U^1 .

Proof. Firstly, equation (5) is reduced to an equivalent system

$$0=u^0, \ \dot{u}^1=Su^1, \quad u^1=Pu, \ u^0=u-u^1. \qquad \text{......................(7)}$$

Secondly, for the second equation in (7) for any $u_0^1 \in U^1$ there exists a unique solution $u^1(t) = e^{tS}u_0^1$ to the problem $\lim_{t\to 0+} u^1(t) = u_0^1$, where

$$e^{tS} = \frac{1}{2\pi i}\int_\Gamma (\mu I - S)^{-1} e^{\mu t} d\mu, \quad t \in R_+,$$

Thus, for any $m \in R$ and $q \in R_+$ the phase space of Eq. (5) is the subspace

$$U^1 = \begin{cases} U, \text{ если } \lambda_k \text{не корень } Q_n(\lambda) \text{ при всех } k \in N; \\ \{u \in U{:}u_k = 0, \text{ если } \lambda_k \text{ – корень } Q_n(\lambda)\}. \end{cases}$$

4. Analytical solution of the Kahn-Hillard model

Let U and F be quasi-Banach spaces of sequences, and let $L \in L(U;F)$ and $M \in Cl(U;F)$ be operators. Then the operator M is strongly $(L,0)$-sectorial. Consider the weakened (in the sense of S.G. Krein) Showalter–Sidorov problem

$$\lim_{t\to 0+} P(u(t) - u_0) = 0 \text{ ....(8)}$$

For a linear nonhomogeneous Sobolev-type evolution equation

$$L\dot{u} = Mu + f, \text{ ....(9)}$$

Where a vector function $f{:}[0,\tau] \to U$, $f = f^0 + f^1$, $f^1 = Qf$, $f^0 = f - f^1$, will be defined below $\tau \in R_+$

Theorem 6. For any vector function $f = f(t)$ such that $f^0 \in C^1((0,\tau);F^0)$ and $f^1 \in C((0,\tau);F^1)$, and any vector $u_0 \in U$, there exists a unique solution $u \in C^1((0,\tau);U)$ to problem (8) for equation (9), moreover, has the form

$$u(t) = -M_0^{-1} f^0(t) + U^t u_0 + \int_0^\tau U^{t-s} L_1^{-1} f^1(s)\, ds. \text{ ....(10)}$$

Proof. Indeed, the fact that $u = u(t)$ satisfies equation (9) and condition (8) is established by direct verification. Uniqueness follows from Theorem 5. The theorem has been proven.

Now consider the Kahn-Hillard equation

$$(\lambda - \Lambda)\frac{\partial u}{\partial t} = \Lambda[F'(u) - \epsilon^2 \Lambda u] + \text{ ....(11)}$$

For any $m, \lambda, \beta \in R$, $\tau, q, \alpha \in R_+$, $u_0 \in U$, $f^0 \in C^1((0,\tau);F^0)$ and $f^1 \in C((0,\tau);F^1)$ there exists a unique solution $u \in C^1((0,\tau);U)$ to problem (8), (9), which has the form

$$u(t) = -M_0^{-1} f^0(t) + U^t u_0 + \int_0^\tau U^{t-s} L_1^{-1} f^1(s)\, ds.$$

Here

$$F^0 = \begin{cases} \{0\}, \ if \quad \lambda_k \neq \lambda \ \ for\ all\ \ k \in N\,; \\ \{f \in F : f_k = 0, \ k \in N \setminus \{\ell : \lambda_\ell = \lambda\}\}; \end{cases}$$

$$F^1 = \begin{cases} F, \ if \quad \lambda_k \neq \lambda \ \ for\ all\ \ k \in N\,; \\ \{f \in F : f_k = 0, \lambda_k = \lambda\}; \end{cases}$$

$$M_0^{-1} = \begin{cases} \mathrm{O}, \ if \quad \lambda_k \neq \lambda \ \ for\ all\ \ k \in N\,; \\ \sum\limits_{k \in N : \lambda_k = \lambda} (\alpha\lambda_k^2 + \beta\lambda_k)^{-1} e_k. \end{cases}$$

$$U^t = \begin{cases} \sum\limits_{k=1}^{\infty} e^{\mu_k t} <\cdot, e_k> e_k, & \text{if } \lambda_k \neq \lambda \ for \ \text{all } k \in N\,; \\ \sum\limits_{k \in N : k \neq \ell} e^{\mu_k t} <\cdot, e_k> e_k, & exists \ \ \ell \in N : \lambda_\ell = \lambda. \end{cases},$$

where $\mu_k = (\alpha\lambda_k^2 + \beta\lambda_k)(\lambda - \lambda_k)^{-1}$.

$$L_1^{-1} = \begin{cases} \sum\limits_{k=1}^{\infty} (\lambda - \lambda_k)^{-1} <\cdot, e_k> e_k, & \text{if } \lambda_k \neq \lambda \ for\ all\ \ k \in N\,; \\ \sum\limits_{k \in N : k \neq \ell} (\lambda - \lambda_k)^{-1} <\cdot, e_k> e_k, & exists \ \ell \in N : \lambda_\ell = \lambda. \end{cases}$$

5. Conclusion

We consider polynomials in the Laplace quasi-operator as operators and and obtain conditions under which holomorphic degenerate semigroups of operators arise in quasi-Banach spaces of sequences, in other words, we prove the first part of the generalization of the Solomyak–Yosida theorem to quasi-Banach spaces of sequences. The phase space of the homogeneous equation is constructed also a "quasi-Banach" analogue of the homogeneous Dirichlet problem in a bounded domain with a smooth boundary for the Kahn-Hillard equation.

Compliance with ethical standards

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