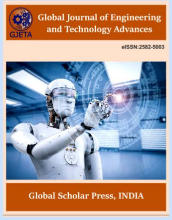
Global Journal of Engineering and Technology Advances

eISSN: 2582-5003

Cross Ref DOI: 10.30574/gjeta

Journal homepage: https://gjeta.com/



(Research Article)

Check for updates

Solution of telegraph equation by double Sumudu-Kamal transform

Mona Hunaiber *

Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen.

Global Journal of Engineering and Technology Advances, 2024, 19(01), 062–067

Publication history: Received on 16 February 2024; revised on 31 March 2024; accepted on 02 April 2024

Article DOI: https://doi.org/10.30574/gjeta.2024.19.1.0049

Abstract

In this paper, we have presented the combination of Sumudu transform and Kamal transform to produce new double integral transform, this double transform called the Sumudu-Kamal transform (SKT). We applied double Sumudu-Kamal transform to solve the linear telegraph equation.

Keywords: Telegraph equation; Double Sumudu-Kamal transform; Integral transform; Sumudu transform; Kamal transform.

1. Introduction

The subject of partial differential equations is one of the most important topics in mathematics, physics, engineering and other sciences. Also, it is significant to know methods to solve such partial differential equations. One of most prevalent method to solve partial differential equations is the integral transforms method [1].

Integral transform has played an important role in solving ordinary and partial differential equations and integral equations, through transform these equations to algebraic equations [2]. The kinds of integral transforms have many applications in various fields of mathematical sciences and engineering such as chemistry, physics, mechanics, acoustic, etc.,[3]. Many phenomena, sciences, processes of engineering and real life can be expressed mathematically and solved by using integral transforms. In previous years, great interest has been given to deal with the double integral transforms such as double Laplace transform, double Sumudu transform, double Laplace-Aboodh transform, double Laplace-Shehu transform, double complex EE transform, etc., for example see [3-7]. These transforms have many applications in various fields of mathematical sciences and engineering.

The telegraph equation is a partial differential equation developed by Oliver Heaviside in 1880. The telegraph equation consisted wave equation in one dimensional and it has applications in solution of communication system problems including the transmission of signals from one point to another [4].

The aim of this work is to solve the linear telegraph equation with initial and boundary conditions by double Sumudu-Kamal transform.

1.1 Definition

The double Sumudu-Kamal transform of the function $\phi(\gamma,\tau)$ for all variables $\gamma,\tau > 0$ is denoted by $S_\gamma K_\tau\left[\phi(\gamma,\tau)\right] = \Phi(\mu,\vartheta)$ and defined by

* Corresponding author: Mona Hunaiber.

Copyright © 2024 Author(s) retain the copyright of this article. This article is published under the terms of the Creative Commons Attribution Liscense 4.0.

$$S_\gamma K_\tau\left[\phi(\gamma,\tau)\right] = \Phi(\mu,\vartheta) = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,\phi(\gamma,\tau)\,d\gamma d\tau, \tag{1}$$

provided the integral exists [8].

The inverse double Sumudu-Kamal transform of the function $\Phi(\mu,\vartheta)$ is defined by

$$\phi(\gamma,\tau) = S_\gamma^{-1}K_\tau^{-1}[\Phi(\mu,\vartheta)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{\gamma}{\mu}}\left(\int_{\beta-i\infty}^{\beta+i\infty} e^{\frac{\tau}{\vartheta}}\,\Phi\left(\mu,\frac{1}{\vartheta}\right)d\vartheta\right)d\mu, \tag{2}$$

where α and β are real constants [8].

2. The Double Sumudu-Kamal Transform of Some Basic Functions

(1). $S_\gamma K_\tau[1] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,d\gamma d\tau = \vartheta.$

(2). $S_\gamma K_\tau[\gamma\tau] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,\gamma\tau\,d\gamma d\tau = \mu\vartheta^2.$

(3). $S_\gamma K_\tau[\gamma^n\tau^m] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,\gamma^n\tau^m\,d\gamma d\tau = n!\,m!\,\mu^n\vartheta^{m+1}, \quad n,m\in\mathbb{N}.$

(4). $S_\gamma K_\tau[\gamma^n\tau^m] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,\gamma^n\tau^m\,d\gamma d\tau = \Gamma(n+1)\mu^n\Gamma(m+1)\vartheta^{m+1}, \quad n,m>-1.$

(5). $S_\gamma K_\tau\left[e^{\alpha\gamma+\beta\tau}\right] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}e^{\alpha\gamma+\beta\tau}\,d\gamma d\tau = \frac{\vartheta}{(1-\alpha\mu)(1-\beta\vartheta)}.$

(6). $S_\gamma K_\tau[sin(\alpha\gamma+\beta\tau)] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,sin(\alpha\gamma+\beta\tau)\,d\gamma d\tau = \frac{\vartheta(\alpha\mu+\beta\vartheta)}{(1+\alpha^2\mu^2)(1+\beta^2\vartheta^2)}.$

(7). $S_\gamma K_\tau[cos(\alpha\gamma+\beta\tau)] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,cos(\alpha\gamma+\beta\tau)\,d\gamma d\tau = \frac{\vartheta-\alpha\beta\mu\vartheta^2}{(1+\alpha^2\mu^2)(1+\beta^2\vartheta^2)}.$

(8). $S_\gamma K_\tau[sinh(\alpha\gamma+\beta\tau)] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,sinh(\alpha\gamma+\beta\tau)\,d\gamma d\tau = \frac{\vartheta(\alpha\mu+\beta\vartheta)}{(1-\alpha^2\mu^2)(1-\beta^2\vartheta^2)}.$

(9). $S_\gamma K_\tau[cosh(\alpha\gamma+\beta\tau)] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,cosh(\alpha\gamma+\beta\tau)\,d\gamma d\tau = \frac{\vartheta+\alpha\beta\mu\vartheta^2}{(1-\alpha^2\mu^2)(1-\beta^2\vartheta^2)}.$

2.1 Theorem [8]. Let $\boldsymbol{\phi(\gamma,\tau)}$ be a continuous function in every finite intervals $(\mathbf{0},\boldsymbol{M})$ and $(\mathbf{0},\boldsymbol{N})$ and of exponential order $\boldsymbol{e^{(\alpha\gamma+\beta\tau)}}$, and $\boldsymbol{S_\gamma K_\tau[\phi(\gamma,\tau)] = \Phi(\mu,\vartheta)}$ then:

(i). $S_\gamma K_\tau\left[\frac{\phi\partial(\gamma,\tau)}{\partial\gamma}\right] = \frac{1}{\mu}\left[\Phi(\mu,\vartheta) - K\left[\phi(0,\tau)\right]\right].$

Proof (i):

$$S_\gamma K_\tau\left[\frac{\phi\partial(\gamma,\tau)}{\partial\gamma}\right] = \frac{1}{\mu}\int_0^\infty\int_0^\infty e^{-\left(\frac{\gamma}{\mu}+\frac{\tau}{\vartheta}\right)}\,\frac{\phi\partial(\gamma,\tau)}{\partial\gamma}\,d\gamma d\tau = \frac{1}{\mu}\int_0^\infty e^{-\frac{\tau}{\vartheta}}d\tau\left(\int_0^\infty e^{-\frac{\gamma}{\mu}}\,\phi_\gamma(\gamma,\tau)d\gamma\right).$$

Using integration by parts, let $\omega = e^{-\frac{\gamma}{\mu}},\ d\sigma = \phi_\gamma(\gamma,\tau)d\gamma$, then we get

$$S_\gamma K_\tau\left[\frac{\phi\partial(\gamma,\tau)}{\partial\gamma}\right]=\frac{1}{\mu}\int_0^\infty e^{-\frac{\tau}{\vartheta}}d\tau\left(-\phi(0,\tau)+\frac{1}{\mu}\int_0^\infty e^{-\frac{\gamma}{\mu}}\phi(\gamma,\tau)d\gamma\right)$$

$$=\frac{1}{\mu}\left[\Phi(\mu,\vartheta)-K\left[\phi(0,\tau)\right]\right].$$

(ii). $S_\gamma K_\tau\left[\frac{\phi\partial(\gamma,\tau)}{\partial\tau}\right]=\frac{1}{\vartheta}\Phi(\mu,\vartheta)-S\left[\phi(\gamma,0)\right].$

(iii). $S_\gamma K_\tau\left[\frac{\partial^2\phi(\gamma,\tau)}{\partial\gamma^2}\right]=\frac{1}{\mu^2}\left[\Phi(\mu,\vartheta)-K\left[\phi(0,\tau)\right]-\mu K\left[\phi_\gamma(0,\tau)\right]\right].$

(iv). $S_\gamma K_\tau\left[\frac{\partial^2\phi(\gamma,\tau)}{\partial\tau^2}\right]=\frac{1}{\vartheta^2}\Phi(\mu,\vartheta)-\frac{1}{\vartheta}S\left[\phi(\gamma,0)\right]-S\left[\phi_\tau(\gamma,0)\right].$

By same method of prove (i), we can prove (ii), (iii) and (iv).

3. Application of The Double Sumudu-Kamal Transform for Telegraph Equation

In this section, we applying the double Sumudu-Kamal transform to solve the telegraph equation. Let the nonhomogeneous telegraph equation in two independent variables (γ,τ) be in the form

$$C^2\phi_{\gamma\gamma}(\gamma,\tau)=\phi_{\tau\tau}(\gamma,\tau)+A\phi_\tau(\gamma,\tau)+B\phi(\gamma,\tau)-\psi(\gamma,\tau), \quad (3)$$

with the initial conditions:

$$\phi(\gamma,0)=\hbar_1(\gamma), \qquad \phi_\tau(\gamma,0)=\hbar_2(\gamma), \quad (4)$$

and the boundary conditions:

$$\phi(0,\tau)=\hbar_3(\tau), \qquad \phi_\gamma(0,\tau)=\hbar_4(\tau), \quad (5)$$

where C, A and B are constants and $\psi(\gamma,\tau)$ is the source term.

Using the property of partial derivative of double Sumudu-Kamal transform for Eq.(3), single Sumudu transform for Eq.(4) and single Kamal transform for Eq.(5), then

$$C^2S_\gamma K_\tau\left[\phi_{\gamma\gamma}(\gamma,\tau)\right]=S_\gamma K_\tau\left[\phi_{\tau\tau}(\gamma,\tau)\right]+AS_\gamma K_\tau\left[\phi_\tau(\gamma,\tau)\right]+BS_\gamma K_\tau\left[\phi(\gamma,\tau)\right]+S_\gamma K_\tau[\psi(\gamma,\tau)],$$

$$\frac{C^2}{\mu^2}\left[\phi(\mu,\vartheta)-\hbar_3(\vartheta)-\mu\hbar_4(\vartheta)\right]=\frac{1}{\vartheta^2}\Phi(\mu,\vartheta)-\frac{1}{\vartheta}\hbar_1(\mu)-\hbar_2(\mu)+\frac{A}{\vartheta}\Phi(\mu,\vartheta)-A\hbar_1(\mu)+B\Phi(\mu,\vartheta)-\Psi(\mu,\vartheta),$$

and simplifying, we get

$$\Phi(\mu,\vartheta)=\frac{\mu^2\vartheta^2}{\mu^2(1+A\vartheta)-\vartheta^2(C^2-B\mu^2)}\left[\left(\frac{1+A\vartheta}{\vartheta}\right)\hbar_1(\mu)+\hbar_2(\mu)-\frac{C^2}{\mu^2}\hbar_3(\vartheta)-\frac{C^2}{\mu}\hbar_4(\vartheta)+\Psi(\mu,\vartheta)\right], \qquad (\gamma,\tau)\in\mathbb{R}_+^2, (6)$$

where $S_\gamma K_\tau[\psi(\gamma,\tau)]=\Psi(\mu,\vartheta)$.

Solving this algebraic equation in $\Phi(\mu,\vartheta)$ and taking the inverse double Sumudu- Kamal transform on both sides of Eq.(6), yields

$$\phi(\gamma,\tau)=S_{\gamma}^{-1}K_{\tau}^{-1}\left(\frac{\mu^2\vartheta^2}{\mu^2(1+A\vartheta)-\vartheta^2(C^2-B\mu^2)}\left[\left(\frac{1+A\vartheta}{\vartheta}\right)\hbar_1(\mu)+\hbar_2(\mu)-\frac{C^2}{\mu^2}\,\hbar_3(\vartheta)-\frac{C^2}{\mu}\,\hbar_4(\vartheta)+\Psi(\mu,\vartheta)\right]\right), \qquad (7)$$

which represent the general formula for the solution of Eq.(3) by double Sumudu- Kamal transform method.

Example 3.1. Consider the following telegraph equation

$$\phi_{\gamma\gamma}(\gamma,\tau)=\phi_{\tau\tau}(\gamma,\tau)+\phi_{\tau}(\gamma,\tau)+\phi(\gamma,\tau), \qquad (\gamma,\tau)\in\mathbb{R}_{+}^{2}, \qquad (8)$$

with the initial conditions

$$\phi(\gamma,0)=e^{\gamma}, \qquad \phi_{\tau}(\gamma,0)=-e^{\gamma}, \qquad (9)$$

and the boundary conditions

$$\phi(0,\tau)=e^{-\tau}, \qquad \phi_{\gamma}(0,\tau)=e^{-\tau}. \qquad (10)$$

Solution: Substituting

$$\hbar_1(\mu)=\frac{1}{1-\mu}, \qquad \hbar_2(\mu)=\frac{-1}{1-\mu},$$

$$\hbar_3(\vartheta)=\frac{\vartheta}{1+\vartheta}, \qquad \hbar_4(\vartheta)=\frac{\vartheta}{1+\vartheta},$$

in Eq.(6), we obtain

$$\Phi(\mu,\vartheta)=\frac{\mu^2\vartheta^2}{\mu^2(1+\vartheta)-\vartheta^2(1-\mu^2)}\left[\left(\frac{1+\vartheta}{\vartheta}\right)\frac{1}{1-\mu}-\frac{1}{1-\mu}-\frac{1}{\mu^2}\,\frac{\vartheta}{1+\vartheta}-\frac{1}{\mu}\,\frac{\vartheta}{1+\vartheta}\right]$$

$$\Phi(\mu,\vartheta)=\frac{\mu^2\vartheta^2}{\mu^2(1+\vartheta)-\vartheta^2(1-\mu^2)}\left[\frac{\mu^2(1+\vartheta)^2-\mu^2\vartheta(1+\vartheta)-\vartheta^2(1-\mu)-\mu\vartheta^2(1-\mu)}{\mu^2\vartheta(1-\mu)(1+\vartheta)}\right].$$

By simplifying, we obtain

$$\Phi(\mu,\vartheta)=\frac{\vartheta}{(1-\mu)(1+\vartheta)}. \qquad (11)$$

Taking the inverse double Sumudu-Kamal transform of Eq.(11), we get a solution of Eq.(8)

$$\phi(\gamma,\tau)=e^{\gamma-\tau}. \qquad (12)$$

Example 3.2. Consider the following nonhomogeneous telegraph equation

$$\phi_{\gamma\gamma}(\gamma,\tau)=\phi_{\tau\tau}(\gamma,\tau)+\phi_{\tau}(\gamma,\tau)-5\phi(\gamma,\tau)+2sin(3\gamma+2\tau), \qquad (\gamma,\tau)\in\mathbb{R}_{+}^{2}, (13)$$

with the initial conditions

$$\phi(\gamma,0)=cos3\gamma=\hbar_1(\gamma), \qquad \phi_{\tau}(\gamma,0)=-2sin3\gamma=\hbar_2(\gamma), \qquad (14)$$

and the boundary conditions

$$\phi(0,\tau)=cos2\tau=\hbar_3(\tau), \qquad \phi_{\gamma}(0,\tau)=-3sin2\tau=\hbar_4(\tau). \qquad (15)$$

Solution: Substituting

$$\hbar_1(\mu)=\frac{1}{1+9\mu^2}\,,\qquad \hbar_2(\mu)=-\frac{6\mu}{1+9\mu^2}\,,$$

$$\hbar_3(\vartheta)=\frac{\vartheta}{1+4\vartheta^2}\,,\qquad \hbar_4(\vartheta)=-\frac{6\vartheta^2}{1+4\vartheta^2}\,,$$

$$\Psi(\mu,\vartheta)=-\frac{2\vartheta(3\mu+2\vartheta)}{(1+9\mu^2)(1+4\vartheta^2)}\,,$$

in Eq.(6), we get

$$\Phi(\mu,\vartheta)=\frac{\mu^2\vartheta^2}{\mu^2(1+\vartheta)-\vartheta^2(1+5\mu^2)}\left[\left(\frac{1+\vartheta}{\vartheta}\right)\hbar_1(\mu)+\hbar_2(\mu)-\frac{1}{\mu^2}\,\hbar_3(\vartheta)-\frac{1}{\mu}\,\hbar_4(\vartheta)-\frac{2\vartheta(3\mu+2\vartheta)}{(1+9\mu^2)(1+4\vartheta^2)}\right]$$

$$=\frac{\mu^2\vartheta^2}{\mu^2(1+\vartheta)-\vartheta^2(1+5\mu^2)}\left[\left(\frac{1+\vartheta}{\vartheta}\right)\frac{1}{1+9\mu^2}-\frac{6\mu}{1+9\mu^2}-\frac{1}{\mu^2}\,\frac{\vartheta}{1+4\vartheta^2}+\frac{1}{\mu}\,\frac{6\vartheta^2}{1+4\vartheta^2}-\frac{2\vartheta(3\mu+2\vartheta)}{(1+9\mu^2)(1+4\vartheta^2)}\right]$$

$$=\frac{\mu^2\vartheta^2}{\mu^2(1+\vartheta)-\vartheta^2(1+5\mu^2)}\left[\frac{\mu^2(1+\vartheta)(1+4\vartheta^2)-6\mu^3\vartheta(1+4\vartheta^2)-\vartheta^2(1+9\mu^2)+6\mu\vartheta^3(1+9\mu^2)-2\mu^2\vartheta^2(3\mu+2\vartheta)}{\mu^2\vartheta(1+9\mu^2)(1+4\vartheta^2)}\right].$$

By simplifying, we obtain

$$\Phi(\mu,\vartheta)=\frac{\vartheta}{\mu^2(1+\vartheta)-\vartheta^2(1+5\mu^2)}\left[\frac{(1+5\mu^2)(6\mu\vartheta^3-\vartheta^2)+(1+\vartheta)(\mu^2-6\mu^3\vartheta)}{(1+9\mu^2)(1+4\vartheta^2)}\right]$$

$$=\frac{\vartheta}{\mu^2(1+\vartheta)-\vartheta^2(1+5\mu^2)}\left[\frac{\big(\mu^2(1+\vartheta)-\vartheta^2(1+5\mu^2)\big)(1-6\mu\vartheta)}{(1+9\mu^2)(1+4\vartheta^2)}\right].$$

$$=\frac{\vartheta-6\mu\vartheta^2}{(1+9\mu^2)(1+4\vartheta^2)}\,. \tag{16}$$

Taking the inverse double Sumudu-Kamal transform of Eq.(16), we get a solution of Eq.(13)

$$\phi(\gamma,\tau)=cos(3\gamma+2\tau). \tag{17}$$

4. Conclusion

In Conclusion, the double Sumudu-Kamal transform is a significant transform among all the integral transforms of exponential sort kernels. The author applied this double transform to obtain the solution of linear telegraph equation.

Compliance with ethical standards

Disclosure of conflict of interest

The author declares no conflicts of interest.

References

[1] Al-Aati A, Hunaiber M, Ouideen Y. On Triple Laplace-Aboodh-Sumudu Transform and Its Properties with Applications. Journal of Applied Mathematics and Computation. 2022; 6(3): 290-309.

[2] Elzaki T, Ishag A. Solution of Telegraph Equation by Elzaki-Laplace Transform. African Journal of Engineering & Technology. 2022; 2(8):1-7.

[3] Hunaiber M, Al-Aati A. On Double Laplace-Shehu Transform and its Properties with Applications. Turkish Journal of Mathematics and Computer Science. 2023; 15(2): 218-226.

[4] Abrori M, Sugiyanto, Sari H. Double Laplace Transform Method for Solving Telegraph equation. JP journal of heat and mass transfer. 2019; 17(1): 265-275.

[5] Eltayeb H, Kiliçman A, On Double Sumudu Transform and Double Laplace Transform. Malaysian Journal of Mathematical Sciences. 2010; 4(1): 17-30.

[6] Hunaiber M, Al-Aati A. The Double Laplace-Aboodh Transform and Their Properties with Applications. Global Scientific Journals. 2022; 10(3): 2055-2065.

[7] Issa A, Kuffi E. On the Double Integral Transform (Complex EE Transform) and Their Applications. Ibn Al-Haitham Journal for Pure and Applied Sciences. 2024; 37(1): 429-441.

[8] Hunaiber M, Al-Aati A. Double Sumudu-Kamal Transform with Applications. Albaydha University Journal. 2022; 4(2):1053-1062.