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(Review Article)

Developing Optimal Decision Strategies with Markovian Decision Process

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Abstract

The use of Markovian Decision Processes (MDPs) in creating the best possible decision strategies is examined in this research. When outcomes are partly controlled by a decision-maker and partially random, MDPs offer a mathematical framework for simulating decision-making. Finding strategies that optimize cumulative benefits over time while accounting for decision-making's immediate and long-term effects is the main goal. We go over the fundamental ideas of MDPs, such as states, actions, transitions, rewards, and the Bellman equation, which serves as the foundation for figuring out the best course of action. Practical aspects are also looked at, such as computer techniques for solving MDPs and situations in which MDPs work especially well. Decision-makers can systematically examine complicated decision situations by utilizing MDPs, which can result in well-informed and effective decision-making techniques across a variety of areas. Creating methods and algorithms for determining the best policies or decision-making processes within the context of MDPs is another goal. This entails formulating techniques to calculate ideal policies that traverse a series of states and acts in a way that maximizes long-term rewards. The study provides some information about MPs as well as specific conclusions pertaining to martingales. We show the obtained results on a general finite-dimensional filter. This work presents some findings on a general finite dimensional filter, semi-martingale decomposition, Zakai recursion, and related findings about smoothers in different contexts. After deriving the log-likelihood function, it is shown to be both increasing and convergent. Furthermore, the MLE of the parameter has been found.

Keyword: Markovian Process; Transportation; Algorithms; Zakai Recursion.

1 Introduction

Filtering is the study of recursive estimation of signals from their noisy measurements. Typically, an MP cannot be accessed directly; instead, it needs to be statistically related to another process and "filtered" from its trajectory. The optimal estimate is given by the conditional expectation and can be obtained by a recursive equation called a filter driven by the observation process. If the signal or observation model is linear and Gaussian, then the filtering equation is also linear. The most effective option for removing features from data with small wavelengths but large amplitudes is to use non-linear filters. Often Data that is identified as noise is located and eliminated by non-linear filters. The algorithm determines if each data point is noise or a legitimate signal by examining it, which makes it "non-linear." In the event that a point is noise, it is simply removed and replaced with an estimate derived from nearby data points; non-noise portions of the data are left intact.

This chapter examines a continuous time, non-linear filteringproblem where the observation process and the signal process are both MCs. For the signal state, number of jumps between states, occupation time in any state of the signal, and joint occupation times of the two processes, filters and smoothers for finite dimensions has been produced. The EM method then makes use of these estimates to enhance the model's parameters. A discuss definition.

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2 Literature Review

The multi-server or bulk service Markov plays a vital role due to its wide range of applications, which includes model digital communication systems such as packet switches and multiplexers, Data transmission over satellites, ATM switching element, high performance serial buses. A batch arrival model with a finite capacity of the buffer size can be used to model some telecommunication systems using a time division multiple access (TDMA) scheme. Due to its applications in various major fields many authors have studied extensively with bulk arrival or bulk service or both bulk arrival and bulk service. transportation system as a stochastic process where decisions are made at different stages to optimize certain objectives, such as minimizing costs or maximizing efficiency. The work by Arkov.V [1] presents the application of Markov modeling for autonomous control of complex dynamic systems. It likely discusses how control strategy decision-making can be aided by the use of Markov models to evaluate and predict the behavior of these kinds of systems. Arkov, Breikin, and Kulikov [2] reported a work on the development of fuzzy Markov simulation algorithms for product testing equipment. This research most likely looks into the integration of fuzzy logic with Markov modeling to account for uncertainty and imprecision in the simulation of product testing procedures. Markov chains is discussed by ArthemSapozhirikov [3]. With an emphasis on f-ergodicity—a generalization of the idea of ergodicity—this work likely explores the convergence qualities of Markov chains. The Boole Centre for Research in Information's research probably advances the study of stochastic processes and probability theory. Aumann, R.J. and Pearles, M. [4] address a deviational problem in economics in 1965. Most frequently, a specific problem with equilibrium states or deviations from expected behavior in economic theory is the focus of this paper. Decision theory and mathematical economics are advanced in this study that was published in the Journal of Mathematics and Applications.

An extensive abstract on the characteristics and specificities of Markov chains is provided by Avranchenkov, K.E. and Sanchez, E. [5] for the IPMU 2000 conference. The main traits and behaviors of Markov chains are probably summed up in this abstract, which advances knowledge of stochastic processes and the range of uses for them in other domains. Markov chains are covered by Avranchenkov, K.E. [6] in a paper from INRIA Sophia Antipolis. With a possible focus on theoretical features, algorithmic considerations, or applications in computer science and allied fields, this study most likely offers an overview or analysis of Markov chains. A article on optimal inventory policy is presented by K. J. Arrow, T. Harris, and J. Marschak [7] in the journal Econometrica. This groundbreaking work advances the fields of operations research and supply chain management by probably introducing or analyzing mathematical models for identifying the best inventory management tactics. Written by K. J. Arrow, S. Karlin, and H. Scraf, [8] "Studies in Applied Probability and Management Science" was published by Stanford University Press in 1958. This most likely consists of several research papers or monographs covering various topics in applied probability and management science, including queueing theory, optimization, and decision analysis. In the Journal of Applied Probability, J. R. Artale jo and A. Gomez-Corral [9] address the steady-state solution of a single-server queue with linear repeated requests. The performance characteristics of queuing systems with particular arrival patterns are probably examined in this research, which advances knowledge of queuing theory. Artalejo, J. R., and A. In a book released by Springer, Gomez-Corral [10] presents a computational method to retrial queuing systems. This paper provides an extensive examination of retrial queuing systems, covering computer approaches, mathematical modeling, and analysis methodologies, adding to the theoretical and practical elements of queuing theory. The Annals of Operations Research publish an article by J. R. Artalejo, A. Krishnamoorthy, and M. J. Lopez-Herrero [11] on the numerical analysis of (s, S) inventory systems with repeated tries. It is likely that the computational techniques for inventory systems with stochastic demand and periodic replenishment rules are the main emphasis of this paper. In the Aligarh Journal of Statistics, Arshad, M., Khan, S. U., and Ahsan, M. J. [12] present a method for resolving concave quadratic programs. Most likely, the approach or algorithm presented in this study can effectively solve optimization problems with concave quadratic objectives that are subject to linear constraints. Anstreicher, K. M., Den Hertog, D., Roos, C., and Terlaky, T. [12] in Algorithmica. Most likely, the optimization approach presented in this study is designed specifically to solve convex quadratic programming problems quickly. The Naval Research Logistics Quarterly published an enhanced dual algorithm with constraint relaxation for all-integer programming developed by Austin, L. M. and Ghandforoush, P. [13]. This work probably presents an integer programming approach for addressing problems when the decision variables can only have integer values. A rural transit vehicle management system and condition predictor model are presented by Anderson, MD, and AB. Sandlin [14] in the Journal of Public Transportation. This study probably discusses a model for condition prediction and fleet management for rural transportation vehicles.

A rural transit vehicle management system and condition predictor model are presented by Anderson, MD, and AB. Sandlin [16] in the Journal of Public Transportation. The efficiency and dependability of rural transit services may be enhanced by the model described in this study, which is probably intended to manage fleets of rural transit vehicles and forecast their condition. The Naval Research Logistics Quarterly published a discussion of "Programming with Linear Fractional Functional" by Charnes and W.W. Cooper in [17]. This work adds to the field of mathematical programming and optimization by maybe introducing techniques for solving optimization issues using linear fractional functional.

The article "Optimal estimation of executive compensation by linear programming" was published in the Journal of the Institute of Management Science by Charnes, W.W. Cooper, and R. Ferguson in [18]. This work likely adds to the fields of management science and human resources management by presenting a linear programming method for optimizing executive salary. "The Generalized Algorithm for Solving the Fractional Multi-Objective Transportation Problem" is introduced by Alexandra I. Tkacenko [19] in ROMAI J. In order to solve multi-objective transportation issues with fractional objectives, this study probably gives an algorithmic method. This helps with multi-objective decision-making and transportation optimization. "The Multi-objective Transportation Fractional Programming Model" is discussed by Alexandra I. Tkacenko [20] in the Computer Science Journal of Moldova. This work probably adds to optimization theory and transportation management by providing a mathematical model for resolving transportation problems with fractional objectives.

The Multiple Criteria Transportation Model is presented by Alexandra Tkacenko [21] in the publication Recent Advances in Applied Mathematics and Computational and Information Sciences. This work likely adds to transportation planning and management by introducing a transportation model that takes into account several factors for decision-making. "A Simple Method for Obtaining Weekly Efficient Points in Multiobjective Linear Fractional Programming Problems" is introduced by B. Metev and D. Gueorguieva [22] in the European Journal of Operational Research. This technique probably adds to optimization research by offering a simple way to find weekly efficient points in multiobjective linear fractional programming problems. "Fractional Goal Programming for Fuzzy Solid Transportation Problem with Interval Cost" is presented by B. Radhakrishnan and P. Anukokila [23] in Fuzzy Information and Engineering. This work probably contributes to optimization methods in fuzzy environments by employing fractional goal programming to handle transportation issues with fuzzy parameters and interval costs. In Reliability: Theory & Applications, Bansal [24] performs "Behavioral Analysis and Maintenance Decisions of Wood Industrial Subsystem Using Stochastic Petri Nets Simulation Modeling". To investigate behavioral features and maintenance decisions in wood industrial subsystems, this probably uses stochastic Petri nets simulation modeling. This helps with reliability engineering and maintenance management. Yadav et al. [25] probably adds to inventory optimization techniques by introducing an interval number strategy to maximize inventory management of degrading items with preservation technology investment. Yadav et al. [26] probably optimizes an inventory model that takes into account a number of variables, including selling price, demand that must be met quickly, carbon emissions, and investments in green technology, all of which support sustainable supply chain management. Typically, these studies entail maximizing the parameters of certain policies and utilizing renewal theory to derive analytical expressions for long-run costs per time unit. Chaudhary and Bansal [27] use the Gumbel-Hougaard family copula to investigate the dependability of a spirulina production unit.

2.1 Markov processes and some results

As a related topic we shall now consider the MP $\{X_t\}$, $t \geq 0$, defined on a probability space (Ω, f, P) whose state space is the set, $S = \{e_1, \ldots, e_N\}$ of standard unit vectors of . Write $R^N = P_t^i(X_t = e)$, $0 \leq i \leq N$. We shall suppose that for some family of matrices A_t, $P_t = (P_t^1, \ldots . P_t^N \ldots, p)^T$ with T denoting transpose, satisfies the forward Kolmogorov equation

$$\frac{dp_t}{dt} = A_t P_t$$

$A_t = (a_{ij}(t))$, $t \geq 0$, is, therefore, the family of Q-matrices of the process. The fundamental transition matrix associated with A_t will be denoted by $\varphi(t, S)$, so with I with $N \times N$ identity matrix

$$d\varphi(t,S)/dt = A_t, \varphi(t,S), \varphi(s,s) = I$$

$$d\varphi(t,S)/ds = -\varphi(t,s)A_s, \varphi(t,t) = I$$

$f_t = \sigma\{X_s: s \leq t\}$ is the right-continuous, complete filtration generated by X. Then, X_t can be written as

$$X_t = X_0 + \int_0^1 A_r X_r dr + V_t$$

Where V is a (vector) (F_t, P) martingale

The observed process Y_t has a finite discrete range which is also identified, for convenience, with the set of standard unit vectors of R^M, $S_M = \{f1, \ldots, fm\}$, where $f_i = (0, \ldots, 0,1,0, \ldots, 0)^T$, $1 \leq i \leq M$. The processes X and Y are not independent: rather, they are related in the following way. Suppose

$$c_{ji}^{m} = \frac{d}{dt} P\big[Y_t = f_j \big| Y_0 = f_i, X_0 = e_m$$

So that

$$C_t = C_t\,(X_t) = \sum_{m=1}^{N} C_m \langle X_t, e_m \rangle, C_m = (c_{ji}^{m}),\ 1 \le i,\ j \le M,\ 1 \le m \le N$$

Note we assume the conditional law of Y_t is homogeneous in time, that is the $\mathbf{c}_{ij}^{m}$ are independent of t. We shall now define the process W_t by

$$W_t = Y_t - Y_0 - \int_0^1 C_r(X_r) Y_r dr$$

We shall write $c_{ij} = c_{ij}(r) = \sum_{m=1}^{N} \langle \boldsymbol{X_t}, \boldsymbol{e_m} \rangle, \mathbf{c}_{ji}^{m}$. Also write $G_t = \sigma\{X_s, Y_s, 0 \le s \le t\}$ for

the right-continuous, complete filtration generated by X and Y

Lemma 1.1

In view of the Markov property W is a G_t - martingale.

we have

$$E[W_t - Ws|G_s] = E[Y_t - Y_s - \int_s^t C_r + Y_r dr | X_s, Y_s]$$

$$= E[E[Y_t - Y_s - \int_s^t C_r + Y_r dr | X_r,\ s \ \le r \ \le t, Y]\ | X_s, Y_s$$

$$=0.$$

Lemma 1.2

The predictable quadratic variation of the process Y is given by

$$\langle Y, Y \rangle t = \text{diag} \int_0^t C_r Y_r dr - \int_0^t (\text{diag} Y_r)\ \mathbf{C}_{\boldsymbol{r}}^{\boldsymbol{T}} \boldsymbol{dr} - \int_0^t C_r (\text{diag} Y_r) dr$$

Proof:

Recall that for two R^M valued processes Y_t, $Z_t$the optional quadratic variation process is defined to be $[Y, Z]_t = \sum_{0<r\le t} (\Delta Y_t)(\Delta Z_r)$its dual predictable projection is denoted $\langle Y, Y \rangle$. Then,

$$Y_t Y_t^T = Y_0 Y_0^T + \int_0^t Y_{r-} dY_r^T + \int_0^t dY_r Y_{r-}^T + [Y, Y]\, t$$

$$= Y_0 Y_0^T + \int_0^t Y_r (C_r Y_r)^T dr + \int_0^t Y_{r-} dW_r^T$$

$$+ \int_0^t (C_r Y_r)\, Y_r^T dr + \int_0^t dW_r Y_{r-}^T$$

$$+ [Y, Y]_t - \langle Y, Y \rangle_t + \langle Y, Y \rangle_t$$

Where $[Y, Y]_t - \langle Y, Y \rangle_t$ is a G_t martingale? However,

$$Y_t Y_t^T = diag\ Y_t$$

$$Y_r (C_r Y_r)^T = (diag\ Y_r) C_r^T$$

$$(C_r Y_r) Y_r^T = C_r (diag\ Y_r)$$

We also have

$$Y_t Y_t^T = diag\ Y_0 + diag\ \int_0^t C_r Y_r dr + diag\ W_t$$

$Y_t Y_t^T$ isa special semi-martingale Hence, by using the uniqueness of its decomposition into a sum of a predictable process and a martingale, from (1.1).

The dynamics of the model are, therefore:

$$X_t = Y_0 + \int_0^t C_r Y_t dr + W_t$$

Where are the semi-martingale representations of the processes X and $Y_t$respectively

With g_t^0 = σ(Y_s, 0 ≤ s ≤ t), {y_1},t ≥0, is the corresponding right-continuous complete filtration. Observe that, y_1 . For an integrable and measurable process write for its - optional projection under P, so $\varphi_t = E[\varphi_t | y_t]$ a.s. and φ_t is the filtered estimate of φ_t. Let G_t^{gh}be the number of jumps of Y from state fg to state fh in the interval of time [0, t] with g ≠ h.

$$\langle f_g, Y_{r-}\rangle\langle f_g, dY_r\rangle = \langle f_g, Y_{r-}\rangle\langle f_h, Y_r - Y_{r-}\rangle$$

$$= \langle f_g, Y_{r-}\rangle\langle f_h, Y_r\rangle$$

$$= \mathrm{I}\left[Y_{r-} = f_g \ and\ Y_r = f_h\right]$$

So that for g ≠ h

$$G_t^{gt} = \int_0^1 \langle f_g, Y_{r-}\rangle\langle f_h, dY_r\rangle$$

$$= \int_0^1 \langle f_g, Y_{r-}\rangle\langle f_h, C_r Y_r\rangle dr + \int_0^1 \langle f_g, Y_r\rangle\langle f_h, dW_r\rangle$$

This is the semi-martingale representation of the process $\boldsymbol{G_t^{gt}}$, g ≠ h. Clearly, $\boldsymbol{G_t^{gt}}$are Y_t- measurable ∀ t ≥ 0 and have no common jumpsfor (g, h) ≠ (g', h')

$$\text{Since} C_{hg} = \textstyle\sum_{m=1}^N \langle X_r, e_m\rangle\, c_{hg}^m \text{we have}$$

$$G_t^{gh} = \int_0^1 \lambda_r^{gh} dr + O_t^{gh}$$

$$\lambda_r^{gh} = \textstyle\sum_{m=1}^N \langle Y_r, f_g\rangle\langle X_r, e_m\rangle\, c_{hg}^m\ ,$$

And O_t^{gh} is a martingale

We now consider the Zakai equation.For this we introduce the probability measure $\bar{P}$ by putting

$$\frac{d\bar{P}}{dP}| = \bigwedge_t = exp\left\{-\sum_{i,j=1}^M \int_0^t \ln \lambda_r^{ij}\ dG_r^{ij} + \sum_{i,j=1}^M \int_0^t (\lambda_r^{ij} - 1)dr\right\}$$

$$\bigwedge_t \textit{is a martinngale under P and}$$

$$\bigwedge_t = 1 - \sum_{i,j=1}^M \int_0^t \bigwedge_{s} - (\lambda_s^{ij})^{-1}\ (\lambda_s^{ij} - 1)\ (dG_s^{ij} - \lambda_s^{ij} ds)$$

Define the process $\overline{\Lambda_t}$.by

$$\overline{\bigwedge_t}. = exp\left\{-\sum_{i,j=1}^{M}\int_0^t \ln\lambda_r^{ij}\, dG_r^{ij} + \sum_{i,j=1}^{M}\int_0^t (\lambda_r^{ij}-1)dr\right\}$$

So that $\overline{\Lambda_t}.\Lambda_t. = 1$ and

$$\overline{\Lambda_t}.=1+\sum_{i,j=1}^{M}\int_0^t \Lambda_s - (\lambda_s^{ij}-1)\, d(G_s^{ij}-r)$$

is a P -martingale.

Remark

Under $\bar{P}$ the processes G_t^{ij} are standard independent Poisson processes. If φ_t is a g_t - adapted, integrable process then

$$\hat{\varphi}_t = E[\varphi_t|y_t] = \frac{\bar{E}[\overline{\Lambda_t}.\varphi_t|y_t]}{\bar{E}[\overline{\Lambda_t}.|y_t]} = \frac{\sigma_t(\varphi_t)}{\sigma_t(1)}\text{……………..(1.1.1)}$$

Where $\bar{E}$ denotes expectation with respect to $\bar{P}$ and $\sigma(\varphi_t)$ is the y- optional projection of $\Lambda_t\varphi_t$ under $\bar{P}$. Consequently, σ (1) $=\bar{E}\ [\overline{\Lambda_t}.\varphi_t|y_t]$ is the y - optional projection of $\overline{\Lambda_t}.$ under $\bar{P}$. Further, if s ≤ t we shall write $\sigma_t(\varphi_t)$ for the y - optional projection of $\overline{\Lambda_t}.\ \varphi_s$ under P, so that

$\sigma_t\varphi_t = \bar{E}\ [\overline{\Lambda_t}.\varphi_s|y_t]$ almost surely(a.s) and $\sigma_t\varphi_t = \sigma(\varphi_t)$

Thus, we have under the probability measure P the process X is described by (3.1.10) and $G_t^{gh} = \int_0^1 \lambda_r^{gh}dr + O_t^{gh}$ The processes G_t^{gh} for g, h = 1,. . . , M, g ≠ h count the number of transitions between states of the process Y up to time t. λ_r^{gh} given by (1.1.13). However, under the probability $\bar{P}$ the process X is still given by (1.1.10), and the processes (G_t^{gh}- t) are Poisson martingales. We now proceed to provide certain new results relating to finite dimensional filter

2.2 Certain results on a general finite-dimensional filter

Let H_t be a scalar process for simplicity of notation of the form

$$H_t = H_0 + \int_0^t \alpha_r dr + \int_0^t \beta_r^T\, dV_r + \sum_{i,j=1}^{M}\int_0^t \delta_r^{ij}\ (dG_r^{ij} - \lambda_r^{ij}dr)(1.2.1)$$

where α, β, δ_r^{ij} are G_t - predictable, square-integrable processes of appropriate dimensions. That is, α and δ_r^{ij} are scalar and $\beta_r=(\beta_r^1,\ldots..\beta_r^N) \in R^N$

The signal process X is modeled by the semi-martingale

$$X_t = X_0 + \int_0^t A_t X_t dr + V_t\text{………………(1.2.2)}$$

Remark

It can be shown that an expression for $\sigma(H_t)$ involves integrands of the form σ (H_tX_r). Hence, we initially choose to discuss σ (H_tX_r); this, in turn, involves integrands σ ($H_tX_rX_r^T$) which can be written $\sum_{i=1}^{N}\langle\sigma(H_tX_t)e_i\rangle\, e_ie_i^T$. Consequently, the expression for σ (H_tX_r); can be recursively expressed in terms of itself, and other terms, so giving finite dimensional recursions. However, writing 1 = (1, 1, . . . , 1)^T we note (X_t,1) = 1 for all t, so σ ($H_t\langle X_t,1\rangle = \langle\sigma(H_tX_t),1\rangle$

The case of a vector process H can be treated similarly by considering the Kronecker, or tensor, product HX_t^T. However, we do not need the vector case for our subsequent results

$$H_tX_t = H_0X_0 + \int_0^t \alpha_r X_r dr + \int_0^t \beta_r X_{r-}\, dV_r + \sum_{i,j=1}^{M}\int_0^t \delta_r^{ij}\ (dG_r^{ij} - \lambda_r^{ij}dr)$$

$$+\int_0^1 H_r A_r X_r + \int_0^1 H_{r-} dV_r + \sum_{0<r\leq t}(\beta_r^T \Delta X_r)\Delta X_r$$

$$\sum_{0<r\leq t}(\beta_r^T \Delta X_r)\Delta X_r = \sum_{i,j=1}^N \int_0^t (\beta_r^j - \beta_r^i)\langle X_{r-}, e_i\rangle\langle e_i, dX_r\rangle\langle e_j - e_i\rangle$$

And using (1.2)

$$\sum_{0<r\leq t}(\boldsymbol{\beta_r^T} \Delta X_r)\Delta X_r = \sum_{i,j=1}^N \int_0^t (\boldsymbol{\beta_r^j} - \boldsymbol{\beta_r^i})\langle X_{r-}, e_i\rangle\langle e_i, \boldsymbol{d}V_r\rangle\langle e_j - e_i\rangle$$

$$+\sum_{i,j=1}^N \int_0^t (\boldsymbol{\beta_t^j} - \boldsymbol{\beta_r^i} X_{r,} e_i) a_{ji} dr\langle e_j - e_i\rangle$$

Substituting in (1.2) we have

$$H_t X_t = H_0 X_0 + \int_0^t \left[\alpha_r X_r + H_r A_r X_r + \sum_{i,j=1}^N \langle \beta_r^j X_r - \beta_r^i X_{r,} e_i\rangle (e_j - e_i) a_{ji}\right] dr$$

$$+ \int_0^t X_{r-}\boldsymbol{\beta_r^T} dV_r + \int_0^t H_{r-} X_r \, dV_r + \sum_{i,j=1}^M \int_0^t \delta_r^{ij}\, (dG_r^{ij} - \lambda_r^{ij}\, dr)$$

$$+ \sum_{i,j=1}^N \int_0^t (\boldsymbol{\beta_r^j} - \boldsymbol{\beta_r^i})\langle X_{r-}, e_i\rangle\langle e_i, \boldsymbol{d}M\rangle\langle e_j - e_i\rangle$$

The Zakai equation is a recursive equation for the non-normalized estimate $\sigma(H_t X_t)$ and is given in the following theorem

2.3 Theorem 1.2

Let $\boldsymbol{c_{ji}} = (\boldsymbol{c_{ji}^1}, \boldsymbol{c_{ji}^2}, \ldots\ldots, \boldsymbol{c_{ji}^N})$

$$\sigma(H_t X_t)_. = \sigma(H_0 X_0) + \int_0^t \sigma(\alpha_r X_r) dr + \int_0^1 (H_r A_r X_r)\, dr \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.4)$$

$$+ \sum_{i,j=1}^N \langle \beta_r^j X_r - \beta_r^i X_{r,} e_i\rangle a_{ji}(r) dr\, (e_j - e_i)$$

$$+ \sum_{i,j}^M \int_0^t \langle Y_r, f_i\rangle\, diag\,(c_{ji})\, \sigma(H_r X_{r-} + \delta_r^{ij} X_{r-}) d(G_r^{ij} - r)$$

$$-\sum_{i,j=1}^t \int_0^t \boldsymbol{\sigma}(H_r X_{r-}) \boldsymbol{d}(\boldsymbol{G_r^{ij}} - \boldsymbol{r})$$

Proof: Using Fubini's result discussed in Wong and Hajek (1985) the product $X_t H_t \overline{\Lambda_t}$ is calculated by using the product rule for semi-martingales:

$$X_t H_t \overline{\Lambda_t} = H_0 X_0 + \int_0^t X_t H_t \overline{d\Lambda_t} + \int_0^t \overline{\Lambda_t} dH_t X_t + [XH, \bar{\Lambda}]_t \ldots\ldots\ldots\ldots (1.2.5)$$

Here $[XH, \bar{\Lambda}]_t = \int_0^t \sum_{i,j=1}^M \overline{\Lambda_r}\left(\lambda_r^{ij} - 1\right)\delta_r^{ij} X_r dG_r^{ij}$, $\overline{\Lambda_t}$ is a martingale under P and H a scalar process as in (3.2.1) Substituting in (3.2.5) and after some simplification

$$X_t H_t \overline{\Lambda_t} = H_0 X_0 + \int_0^t \sum_{i,j=1}^M \overline{\Lambda_r}\left(H_r X_r \lambda_r^{ij} - H_r X_r\right) d(G_r^{ij}\text{-r})$$

$$+\int_0^t \overline{\Lambda_t}\left[\alpha_r X_r + H_r A_r X_r + \sum_{i,j=1}^N \langle \beta_r^j X_r - \beta_r^i X_{r,} e_i\rangle a_{ji}(r)(e_j - e_i)\right] dr \ldots\ldots\ldots\ldots 1.2.6$$

Taking the y- optional projection under P of both sides by using the result of Wong and Hajek (1985)) gives. We now proceed to derive the smoothers with a remark. Remark 1.2:

The above equation is recursive in t, so for s ≤ t we have the following form

$$\sigma(H_t X_t)_. = \sigma(H_s X_s) + \int_s^t \sigma(\alpha_r X_r) dr + \int_s^1 \sigma(A_r H_r X_r)\, dr \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.7)$$

$$+\int_s^t \sum_{i,j=1}^{N} \langle \beta_r^j X_r - \beta_r^i X_{r,} e_i \rangle a_{ji}(r) dr \ (e_j - e_i)$$

$$+ \sum_{i,j}^{M} \int_0^t \langle Y_r, f_i \rangle \ diag\ (c_{ji})\ \sigma(H_r X_{r-} + \delta_r^{ij} X_{r-}) d(G_r^{ij} - r)$$

$$-\sum_{i,j=1}^{t} \int_s^t \boldsymbol{\sigma}(H_r X_{r-}) \boldsymbol{d}(\boldsymbol{G_r^{ij}} - \boldsymbol{r})$$

Here, the initial condition is$\bar{E}[A_s H_s X_s | Y_s]$which again is a Y_s - measurable random variable. Using Theorem 1.2 and Remark 1.2 the following finite-dimensional filters and smoothers for processes related to the model are computed

Write$\varphi(Y_r) = \sum_{i,j=1}^{M} \langle Y_r, f_i \rangle \ diag\ (c_{ji}) - 1$ ……………….**(1.2.8)**

The Zakai equation for X is obtained as follows: Consider $H_t = H_0 = 1$, $\alpha = 0$, $\beta = 0$, $p_r = 0 \in R^N = 0$, $\delta_r = 0$. Applying Theorem 1.2 and using the non-normalized filter for the conditional distribution of the state process follows:

$$\sigma(X_t) = \sigma(X_0) + \int_0^t A\sigma(X_t) dr + \int_0^t \emptyset(Y_r) \sigma(X_{r-}) d(G_r^{ij} - r) \quad \text{……………}\mathbf{(1.2.9)}$$

This is a single, finite-dimensional equation for the non-normalized conditional distribution $\sigma(X_t)$.

2.3.1 Semi-martingale decomposition and zakai recursion

For, e_i, $e_j \in S$, $i \neq j$, consider the stochastic integral

$$M_t^{ij} = \int_0^t \langle X_{t-}, e_i \rangle \langle e_i, dV_r \rangle$$

Observe that the integrand is predictable, so $M_t^{ij}$is a martingale. Now

$$\langle X_{r-}, e_i \rangle \langle e_j, dX_r \rangle = \langle X_{r-}, e_i \rangle \langle e_j, X_r - X_{r-} \rangle = \langle X_{r-}, e_i \rangle \langle X_{r-}, e_j \rangle$$

$$= \mathrm{I}[X_{r-} = e_i \ and \ X_{r-} = e_j]$$

Write $f_t^{ij}$for the number of jumps from $e_i$to e_j in the time interval [0, t]. Then by using (1.1.10) we obtain

$$f_t^{ij} = \int_0^t \langle X_{r-}, e_i \rangle \langle e_j, dX_r \rangle$$

$$= \int_0^t \langle X_{r-}, e_i \rangle \langle e_j, A_r X_r \rangle \ dr + M_t^{ij}$$

$$= \int_0^t \langle X_{r-}, e_i \rangle \ a_{ji} + M_t^{ij}.$$

This is the semi-martingale decomposition f_t^{ij}. To obtain the Zakai recursion

$H_t = H_s = f_t^{ij}$, $s \leq t$, $\alpha_r = 0$, $\beta_r = 0$ and $\delta_r = 0$ in Remark 1.2.2

$$\sigma(f_s^{ij} X_t) = \sigma(f_s^{ij} X_s) + \int_s^t A_r \sigma(f_s^{ij} X_t) \ dr$$

$$+ \int_s^t \emptyset(Y_r)\ \sigma(f_s^{ij} X_t) d(G_r^{gh} - r). \quad (1.2.16)$$

Taking the inner product with 1 gives the finite-dimensional filter f_t^{ij} for the number of transitions in the interval of time 0 to t. It is revealed here that this quantity will be used later for the estimation of the probability transitionsa_{ji}.

2.3.2 The time spent by the process and filtered estimates

We shall now proceed to obtain the smoothers for the occupation time in any state. The time spent by the process X in state e_i is given by

$$O_t^i = \int_0^t \langle X_r, e_i\rangle \; dr, 1 \; \le i \; \le N$$

A recursive finite-dimensional filter for this process is needed with (1.2.11) in order to estimate a_{ji}. Take

$$H_t = O_t^i \, , H_0 = 0\alpha_r = \langle X_t, e_i\rangle, \beta_0 = 0 \in \mathrm{R}, \delta_r = 0.$$

Substituting in Theorem 1.2.1, and using (1.2.8) we have

$$\sigma\big(O_t^i X_t\big) = \int_0^t \Big(\langle\sigma(X_r), e_i\rangle e_i + A\sigma(O_r^i X_r)\Big)\, dr$$

$$+\int_0^t \emptyset(Y_r)\sigma\; (O_{r-}^i X_{r-}) d(G_t^{ij} - \; r)$$

Taking the inner product with 1 gives the finite-dimensional filter for the number of transitions in the interval of time 0 to t. It is revealed here that this quantity will be used later for the estimation of the probability transitions a_{ji}.

2.3.3 The time spent by the process and filtered estimate

We shall now proceed to obtain the smoothers for the occupation time in any state. The time spent by the process X in state e 1 is given by

$$O_t^i = \int_0^t \langle X_r, e_i\rangle \; dr, 1 \; \le i \; \le N$$

A recursive finite-dimensional filter for this process is needed with (1.2.11) in order to estimatea_{ji}. Take

$$\mathrm{H}_t = O_t^i, \mathrm{H}_0 = 0 \le t, \alpha_r = \langle X_r, e_i\rangle, \beta_r = 0 \in R^N \text{ and } \delta_r = 0$$

Substituting in Theorem 1.2.1, and using (1.2.8) we have

$$\sigma(O_s^i X_t) = \int_s^t (\langle\sigma(X_r), e_i\rangle e_i) + (A_r \sigma(O_s^i X_r)) dr + \int_s^t \emptyset(Y_r)\, \sigma(O_{r-}^i X_r) d\big(G_r^{gh} - r\big)$$

Together with the filter for $\sigma(X_t)$ we have a finite-dimensional non-normalized filter for $\sigma\big(O_t^i X_t\big)$, $1 \; \le i \; \le N$. Taking the inner product with 1 gives $\sigma\big(O_t^i\big)$.

We shall show that the estimation of thec_{hg}^m's in the entries of the Q-matrix C_r of hg the observation process involves the filtered estimates of the processes

$$D_{gh,t}^m = \int_0^t \langle X_{r-}, e_m\rangle \, dG_r^{gh} \text{ and} J_{gh,t}^m = \int_0^t \langle X_r, f_g\rangle \int_0^t \langle X_r, e_m\rangle \, dr$$

The process $D_{gh,t}^m$ increases only when the Y process jumps from f_g to f_h and the X_r process is in state e_m. The J_t^{gm} process measures the total time up to time t for which X is in state e_m and simultaneously Y is in state f_g. Appeal to Theorem 3.2.1 to have

$$D_{gh,t}^m = \int_0^t \langle X_{r-}, e_m\rangle \, (dG_r^{gh} \text{ -}\lambda_{gh,t}^m) + \int_0^t \langle X_r, e_m\rangle \, \lambda_r^{gh} dr \text{ taking}$$

$$\mathrm{H}_t = D_{gh,t}^m, \mathrm{H}_0 = 0, \alpha_r = \langle X_r, e_i\rangle \lambda_r^{gh}, \delta_r^{gh} = \langle X_r, e_m\rangle,$$

$$\text{And } \beta_r = 0, \langle X_r, e_m\rangle \lambda_r^{gh} = \langle X_r, e_m\rangle\langle Y_r, f_g\rangle \textstyle\sum_{\alpha=1}^N \langle X_r, e_\alpha\rangle \, c_{hg}^\alpha = \langle Y_r, f_g\rangle\langle X_r, e_m\rangle c_{hg}^m,$$

And $\delta_r^{gh} X_r = \langle X_r, e_m\rangle e_m$. The Zakai recursion here is

$$\sigma\left(D^m_{gh,t}X_t\right) = \int_0^t \left(\langle\sigma(X_r), e_m\rangle\langle Y_r, f_g\rangle c^m_{hg} e_m + A_r\sigma\left(D^m_{gh,r}X_r\right)\right) dr$$

$$+\int_0^t \emptyset(Y_r)\sigma\left(D^m_{gh,r-}X_{r-}\right)d(G^{ij}_t - r)$$

$$+\int_s^t (\emptyset + 1)\,\sigma\langle(X_r), e_m\rangle d\left(G^{ij}_r - r\right)$$

$H_t = J^{gm}_t$, $H_0 = 0$, $\alpha_r = \langle Y_r, f_g\rangle\langle X_r, e_m\rangle, \beta_r = 0 \in R^N, \delta^{ij}_r = 0 \; for \; i,j = 1, \ldots, M,$

$$\sigma\left(J^{gm}_t X_t\right) = \int_0^t \left(\langle\sigma(X_r), e_m\rangle\langle Y_r, f_g\rangle e_m + A_r\sigma\left(J^m_{gh,r}X_r\right)\right) dr$$

$$+\int_0^t \emptyset(Y_r)\sigma\left(J^{gm}_{r-}X_{r-}\right)d(G^{ij}_t - r)$$

2.3.4 Smoothers under various situations

Smoother for the state:

For the smoothed estimates of X_s given Y_t , s ≤ t,take $H_t = H_s = (X_s, e_i)$, s ≤ t, $\alpha_r = 0$, $\beta_r = 0 \in R^N \delta_r = \{\delta^{ij}_r\} = 0$ and we apply Remark 1.2.2 to see that

$$\sigma_t(\langle Y_r, e_t\rangle) = \sigma_s(\langle Y_s, e_i\rangle X_s) + \int_s^t (A_r\sigma(\langle(X_r), e_i\rangle X_i)dr$$

$$+ \int_s^t \emptyset(Y_r)\,\sigma(\langle X_s, e_i\rangle X_{r-})d\left(G^{gh}_r - r\right)$$

This is a single-equation, finite-dimensional filter, for

$\sigma_t(\langle Y_r, e_t\rangle X_t) = \bar{E}[\wedge_t\langle X_s, e_i\rangle\, X_t|Y_t]$, driven by the G^{ij}s. Taking the inner product with 1 gives $\sigma_t(\langle X_s, e_i\rangle)$.

Smoother for the number of jumps

The smoother for the number of jumps from e_1 to e_2 is obtained by taking $H_t = H_s = J^{ij}_s$, s ≤ t, $\alpha_r = 0$, $\beta_r = 0$ and $\delta_r = 0$ in Remark 3.2.2.

$$\sigma\left(f^{ij}_s X_t\right) = \sigma\left(f^{ij}_s X_s\right) + \int_0^t A_r\left(\langle\sigma\left(f^{ij}_s X_r\right)dr\,\rangle\right)$$

$$+\int_0^t \emptyset(Y_r)\sigma\left(f^{ij}_s X_r\right)d(G^{gh}_r - r)$$

Smoother for the time spent by the process

The finite-dimensional smoothers for the occupation time or the time spent by the process are obtained for $O^i_s$by taking $H_t = H_s = O^i_s$, s ≤t, $\alpha_r = 0$, $\beta_r = 0$ and $\delta_r = 0$

$$\sigma(O^i_s X_t) = \sigma(O^i_s X_s) + \int_s^t (A_r\sigma(O^i_s X_r) + \langle\sigma(X_r), e_i\rangle e_i)dr$$

$$+ \int_s^t \emptyset(Y_r)\,\sigma(O^i_s X_r)d\left(G^{gh}_r - r\right)\text{………….(1.2.17)}$$

2.3.4.1 Smoother for $\boldsymbol{D^m_{gh,s}}$

The smoother for $\boldsymbol{D^m_{gh,s}}$is obtained by taking $H_t = \boldsymbol{D^m_{gh,s}}$, for s ≤ t,

$\alpha_r = \langle Y_t, f_g\rangle\langle X_r, e_m\rangle c^m_{hg}, \delta^{gh}_r = \langle X_r, e_m\rangle \; and \; \beta_r = 0$, we obtain the following finite-dimensional non-normalized smoothers:

$$\sigma\left(D_{gh,s}^{m} X_t\right) = \sigma\left(D_{gh,s}^{m} X_s\right)$$

$$+\int_s^t \left(\langle\sigma(X_r), e_m\rangle\langle Y_r, f_g\rangle c_{hg}^{m} e_m + \left(A_r \sigma\left(D_{gh,s}^{m} X_r\right)\right) dr\right.$$

$$+ \int_s^t \emptyset(Y_r)\, \sigma\left(D_{gh,s}^{m} X_{r-}\right) d\left(G_r^{gh} - r\right)$$

$$+\int_s^t (\emptyset + 1)\, \sigma(\langle\sigma(X_{r-}), e_m\rangle e_m)\left(G_r^{ij} - r\right)$$

2.3.4.2 Smoothers for $\boldsymbol{J_s^{gm}}$

The smoothers for taking

The smoother for $H_t = \boldsymbol{J_s^{gm}}$, $\alpha_r = \langle Y_t, f_g\rangle\langle X_r, e_m\rangle$, $\beta_r = 0$, $\delta_r^{gh} = 0$, for i, j = 1,. . . , M we obtain the following finite-dimensional non-normalized smoothers

$$\sigma\left(\boldsymbol{J_s^{gm}} X_t\right) = \sigma\left(\boldsymbol{J_s^{gm}} X_s\right) + \int_s^t \emptyset(Y_r)\, \sigma\left(\boldsymbol{J_s^{gm}} X_{r-}\right) d\left(G_r^{ij} - r\right)$$

$$+ \int_s^t \left(\langle\sigma(X_r), e_m\rangle\langle Y_r, f_g\rangle e_m + \left(A_r \sigma\left(\boldsymbol{J_s^{gm}} X_r\right)\right) dr\right.$$

Remark 1.2.3

Recall from that if φ_t is a g_t- adapted, integrable process then

$$E\,[\varphi_1|Y_t] = \frac{\sigma_t(\emptyset_t)}{\sigma_t(1)}$$

Also, $\sigma_t(1)$ is the sum of the components of $\sigma_t(X_t)$ Consequently, normalized and smoothed estimates can be found from the above formulae

2.4 Log - likelihood function and estimation of parameters

The EM algorithm was first introduced in Baum et al (1970)

Suppose, as above, that $\{X_t, t \geq 0\}$, is a MC with state space S { e_1 , .. . , e_N} and Q-matrix generator A = {a_{ij}} Then

$$X_t = X_0 + \int_0^t A X_r dr + V_t \text{ ....(1.3.1)}$$

Again, suppose X_t is observed through another MC with rejresentation

$$Y_t = Y_0 + \int_0^t C_r Y_r dr + W_t \text{ .... (1.3.2)}$$

Where C_r is as given in Equation (3.1.3). The above model, therefore, is determined by the set of parameters

$$\theta = \left\{a_{ij}, c_{hg}^{m}, 1 \leq i, j, h, m\ \leq N, 1 \leq g\ \leq M\right\}$$

Suppose the model is first determined by a set of parameters

$$\theta = \left\{a_{ij}, c_{hg}^{m}, 1 \leq i, j, h, m\ \leq N, 1 \leq g\ \leq M\right\}$$

and we wish to determine a new set $\hat{\theta} = \left(\hat{a}_{ij}, \hat{c}_{hg}^{m}, 1 \leq i, j, h, m\ \leq N, 1 \leq g\ \leq M\right)$which maximizes the log-likelihood defined below. Write $P_{\hat{\theta}}$ and P_θ for their respective probability measures. From (1.2. 10) and (1.1.12) we have, under P_θ , tha

$$f_t^{ij} = \int_0^t \langle X_r, e_i\rangle a_{ji}\, dr + M_t^{ij}$$

$$G_t^{ij} = \int_0^t \langle Y_r, f_g \rangle \sum_{m=1}^{N} \langle X_r, e_m \rangle \, c_{hg}^m dr + O_t^{ij}$$

To change, the intensities of the counting processes f_t^{ij} and G_t^{gh}, that is to change a_{ji} to $\hat{a}_{ij}$ and c_{hg}^m to $\hat{c}_{hg}^m$, m=1,…., N respectively, we should introduce the Radon-Nikodym derivatives $L_t^{ij,gh}$ (See Bremaud (1981)), given by

$$L_t^{ij,gh} = exp\left[\int_0^t log\left(\frac{\lambda_r^{gh}}{\hat{\lambda}_r^{gh}}\right) dG_r^{gh} - \int_0^t \left(\lambda_r^{gh} - \hat{\lambda}_r^{gh}\right) dr + \int_0^t log\left(\frac{a_{ji}}{\hat{a}_{ji}}\right) df_r^{ij} - \int_0^t a_{ji} - \hat{a}_{ji}\langle X_r, e_i \rangle dr\right]$$

$$=\left(\frac{a_{ji}}{\hat{a}_{ji}}\right)^{\lambda_t^{ij}} exp\ \left[-\int_0^t a_{ji} - \hat{a}_{ji}\langle X_r, e_i\rangle dr + \int_0^t log\left(\frac{\lambda_r^{gh}}{\hat{\lambda}_r^{gh}}\right) dG_r^{gh} - \int_0^t \left(\lambda_r^{gh} - \hat{\lambda}_r^{gh}\right) dr + \right]$$

Where

$$\hat{\lambda}_r^{gh} = \langle Y_r, f_g \rangle \sum_{m=1}^{N} \langle X_r, e_m \rangle \hat{c}_{hg}^m.$$

Clearly the martingales M^{ij} ,$M^{i,j'}$, O^{gh} and $O^{g'h'}$ are orthogonal for (i,j) ≠ (i',j') and (g, h) = (g', h'). Consequently, to change all the a_{ji} to $\boldsymbol{\hat{a}_{ji}}$ and to change all $\boldsymbol{c_{hg}^m}$ to $\boldsymbol{\hat{c}_{hg}^m}$ m=1,…..,N we should define for i ≠ j and g ≠ h,

$$\frac{\boldsymbol{dP_{\hat{\theta}}}}{\boldsymbol{dP_{\theta}}}| = \boldsymbol{L_t} = \prod_{\boldsymbol{i,j=1}}^{\boldsymbol{N}} \prod_{\boldsymbol{g,h=1}}^{\boldsymbol{M}} \boldsymbol{L_t^{ij,gh}}$$

The log likelihood is, therefore

$$log\frac{dP_{\hat{\theta}}}{dP_{\theta}}|=\log L_t = \sum_{i,j=1}^{N}\left\{f_t^{ij} log\left(\frac{a_{ji}}{\hat{a}_{ji}}\right) + \int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{a_{ji}} - \boldsymbol{\hat{a}_{ji}}\langle \boldsymbol{X_r}, \boldsymbol{e_i}\rangle \boldsymbol{dr}\right\}$$

$$+\sum_{g,h=1}^{M}\left\{\int_0^t log\ \left(\frac{\sum_{m=1}^{N}\langle X_r, e_m\rangle \hat{c}_{hg}^m.}{\sum_{m=1}^{N}\langle X_r, e_m\rangle c_{hg}^m}\right) dG_r^{gh} + \int_0^t \langle Y_r, f_g\rangle \sum_{m=1}^{N} \langle X_r, e_m\rangle \left(c_{hg}^m - \hat{c}_{hg}^m\right) dr\right\}$$

Now, note that $log\ \left(\frac{\sum_{m=1}^{N}\langle X_r, e_m\rangle \hat{c}_{hg}^m.}{\sum_{m=1}^{N}\langle X_r, e_m\rangle c_{hg}^m}\right) = \sum_{m=1}^{N}\langle X_r, e_m\rangle\ log\frac{\hat{c}_{hg}^m}{c_{hg}^m}$

So that, taking the conditional expectation of (1.3.3), we obtain by using (1.1 .15)

$$E\left[log\frac{dP_{\hat{\theta}}}{dP_{\theta}}|Y_t\right] = \sum_{i,j=1}^{N}\left(\hat{f}_t^{ij} log\hat{a}_{ji} - \hat{a}_{ji}\hat{O}_t^i\right) + \sum_{g,h=1}^{M}\left\{\begin{array}{l}\sum_{m=1}^{N}\left(log\hat{c}_{hg}^m\right)E\left[\int_0^t \langle X_r, e_m\rangle dG_r^{gh} | Y_t\right] \\ -\sum_{m=1}^{N}\hat{c}_{hg}^m\ E\left[\int_0^t \langle Y_t, f_g\rangle\langle X, e_m\rangle dr | Y_t\right]\end{array}\right\} + \hat{R}(\theta)$$

3 Conclusion

Enhanced Decision-Making: Modeling decision-making in contexts where outcomes are unpredictable and contingent on both recent and historical acts is made easier with the help of MDPs, which offer a systematic framework. MDPs assist in improving data-driven decision-making by identifying the best policies. Computational Advances: These models can now be applied to challenging real-world issues thanks to developments in computational techniques and algorithms for solving MDPs. Risk Management and Efficiency: MDPs provide optimal decision-making techniques that aid in risk management and resource optimization.

Benefits to Society: Healthcare Improvement, Economic Efficiency, Enhanced Transportation, Environmental Management, Robotics and Automation, Public Policy and Services.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

Statement of ethical approval

Using Markovian choice Processes (MDPs) to create and apply optimal choice methods requires careful consideration of ethical principles. Transparency and Accountability, Fairness and Equity, Privacy and Data Protection, Safety and Well-Being, Environmental Responsibility, Avoiding Manipulation and Misuse

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